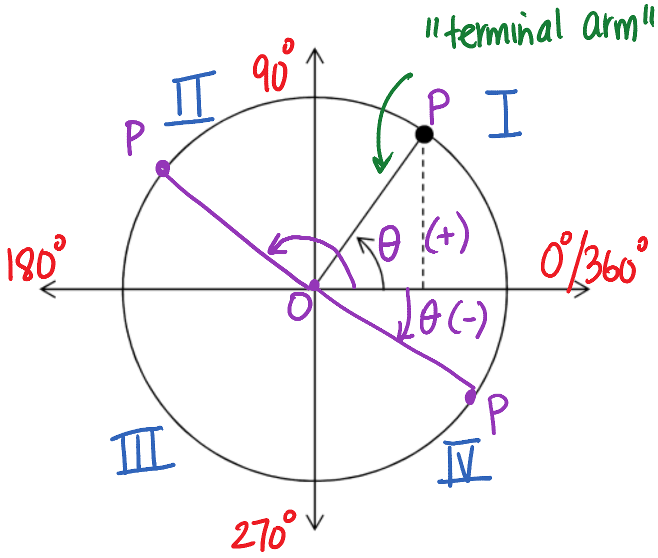
Notes

March-30-16
9:57 AM

2.1: Angles in Standard Position (Day 1)

Angles in Standard Position, $0° \le \theta < 360°$

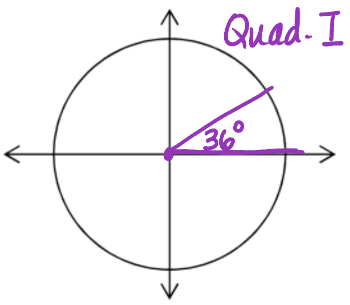


- a circle centered at the **origin:** (0,0)
- "theta"
  Standard position: $\angle\theta$ is the angle that starts at the positive x-axis and ends at the line OP (**terminal arm).**
- if $\angle\theta$ is positive $\rightarrow$ counter clockwise (CCW) (up)
- if $\angle\theta$ is negative $\rightarrow$ clockwise (CW) (down)

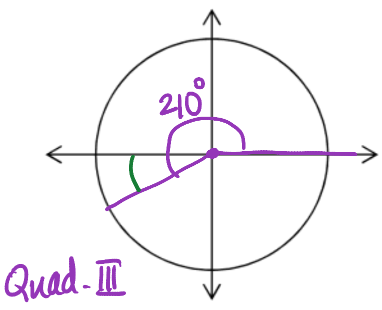
I, II, III, IV, V

Ex 1: Sketch each angle in standard position. State the quadrant in which the terminal arm lies.

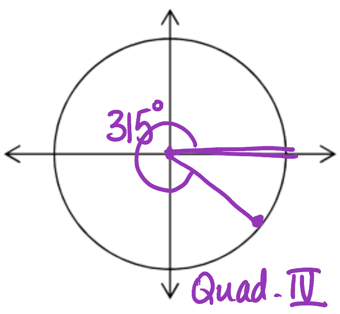
a) 36°



b) 210°



c) 315°



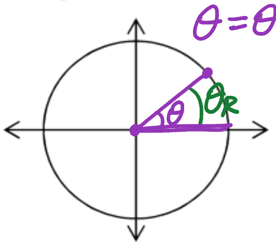
Reference Angles

- For each angle in standard position, there is a corresponding **acute** (<90°) angle called the **reference angle.** (θ_R)

Reference angle (θ_R): the acute angle whose vertex is the origin and whose arms are the terminal arm of the angle and the x-axis. *Short-cut to the x-axis.

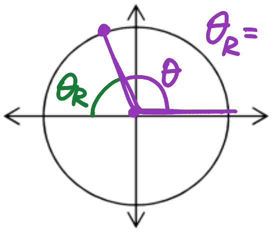
Quadrant 1

$\theta = \theta_R$

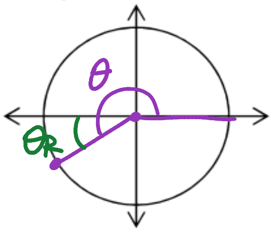


Quadrant 2

$\theta_R = 180° - \theta$

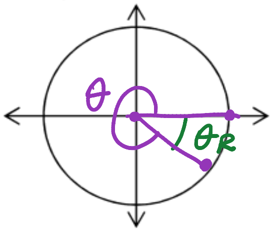


Quadrant 3



$\theta_R = \theta - 180°$

Quadrant 4



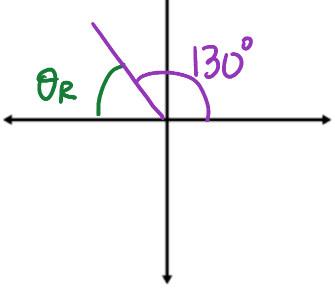
$\theta_R = 360° - \theta$

Ex 2: Determine the reference angle θ_R for each angle θ. Sketch θ in standard position and label the reference angle θ_R.

a) $\theta = 130°$

$\theta_R = 180° - 130°$

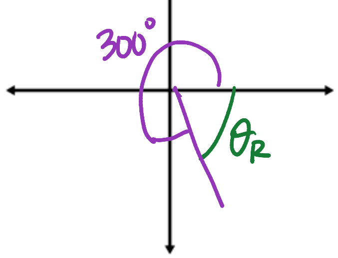
$\boxed{\theta_R = 50°}$



b) $\theta = 300°$

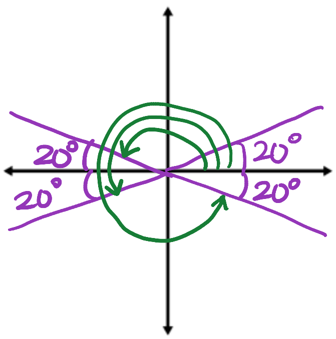
$\theta_R = 360° - 300°$

$\boxed{\theta_R = 60°}$



Ex 3: Determine all of the possible angles in standard position with a reference angle of 20°

$\theta_R = 20°$



QI: $\theta = 20°$

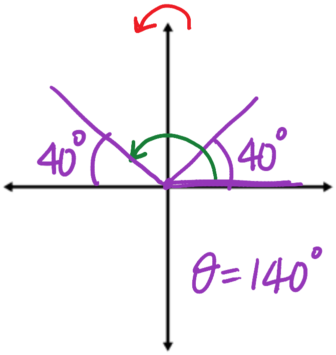
QII: $\theta = 160°$

QIII: $\theta = 200°$

QIV: $\theta = 340°$

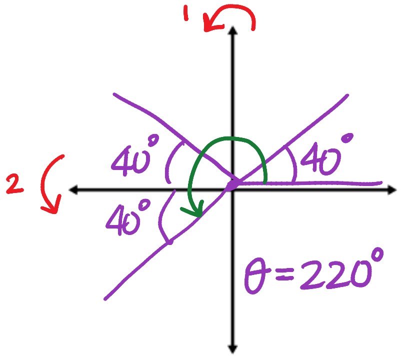
Ex 4: Determine the angle in standard position when an angle of 40° is reflected:

a) in the y-axis



$\theta = 140°$

b) in the y-axis and then in the x-axis



$\theta = 220°$

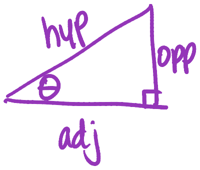
2.1: Angles in Standard Position (Day 2)

Recall:

(sine) $\sin\theta = \dfrac{opposite}{hypotenuse}$

(cosine) $\cos\theta = \dfrac{adjacent}{hypotenuse}$

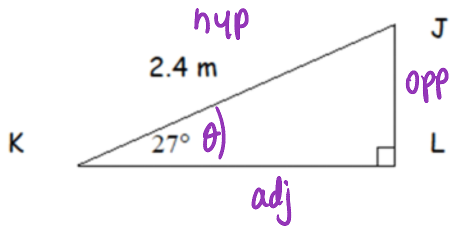
(tangent) $\tan\theta = \dfrac{opposite}{adjacent}$



Just remember:

sinθ	cosθ	tanθ
↓	↓	↓
SOH	CAH	TOA

Ex 1: Solve the following right triangle (determine all sides and angles).



$\angle J = 180° - 27° - 90° = \boxed{63°}$

$JL: \sin 27° = \frac{JL}{2.4}$

$\boxed{JL = 1.1\text{ m}}$

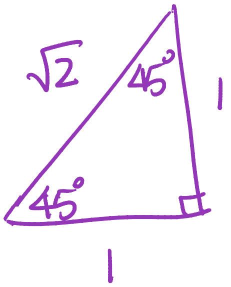
$KL: \cos 27° = \frac{KL}{2.4}$

$\boxed{KL = 2.1\text{m}}$

Special Right Triangles

I. 45, 45, 90 triangle:

Draw a right triangle with angles 45°, 45°, and 90° with the 2 legs being 1 unit long. Determine the length of the hypotenuse. Then determine:



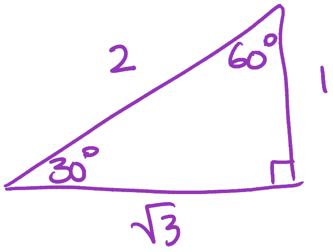
$\sin 45° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\cos 45° = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$\tan 45° = \frac{1}{1} = 1$

II. 30, 60, 90 triangle:

Draw a right triangle with angles 30°, 60° and 90° with a hypotenuse of 2 units and 1 leg of 1 unit. Determine the length of the other leg. Then determine:



$\sin 30° = \frac{1}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}$

$\cos 30° = \frac{\sqrt{3}}{2}$

$\cos 60° = \frac{1}{2}$

$\tan 30° = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\tan 60° = \sqrt{3}$

*** You will need to **memorize** the two special triangles angles and sides.

Ex.2: Using the given ratios and the given quadrant, draw the triangle and find the angle in the standard position circle

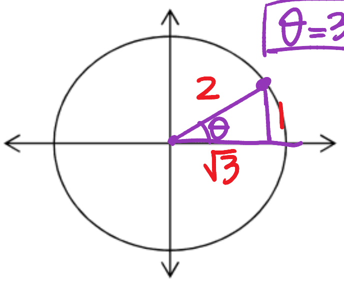
a) $\sin\theta = \frac{1}{2}$, QI

opp

hyp

*special Δ

$\boxed{\theta = 30°}$



b) $\cos\theta = \frac{\sqrt{3}}{2}$, Q IV

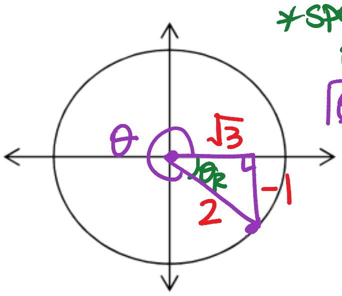
adj

hyp

*special Δ

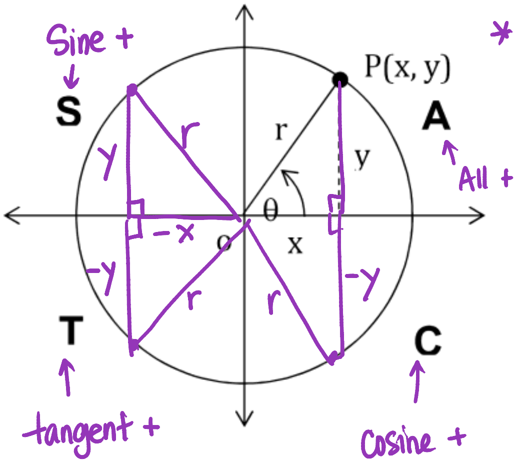
$\theta_R = 30°$

$\boxed{\theta = 330°}$



2.2: Trigonometric Ratios of Any Angle

Given the following **circle** diagram, determine:



*r is always positive

a) the length of r:

$x^2+y^2=r^2$

$r=\sqrt{x^2+y^2}$

b) $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$

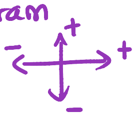
c) $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$

d) $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$

Each trigonometric ratio is positive in exactly 2 quadrants and negative in exactly 2 quadrants. → Why?

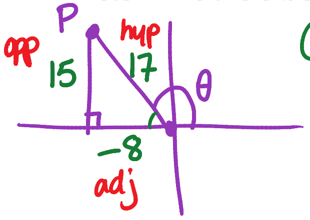
*See Diagram

*Direction



- **Quadrant I (A = ALL)** All 3 ratios are positive.
- **Quadrant II (S = STUDENTS)** Only the Sine ratio is positive.
- **Quadrant III (T = TAKE)** Only the Tangent ratio is positive.
- **Quadrant IV (C = CALCULUS)** Only the Cosine ratio is positive.

Ex 1: The point P(-8, 15) lies on the terminal arm of an angle, θ, in standard position. Determine the exact trigonometric ratios for $\sin\theta$, $\cos\theta$ and $\tan\theta$.



$(-8)^2+(15)^2=r^2$

$289=r^2$

$17=r$

$\sin\theta = 15/17$

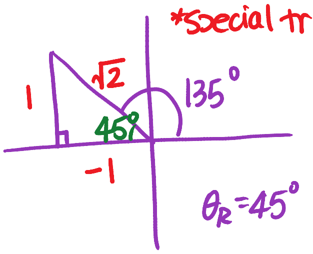
$\cos\theta = -8/17$

$\tan\theta = 15/-8 = -15/8$

Ex 2: Determine the exact value of cos135° without a calculator. (Draw a diagram!)

*Special triangle!

$\theta=135^\circ$



$\theta_R=45^\circ$

Rationalized!

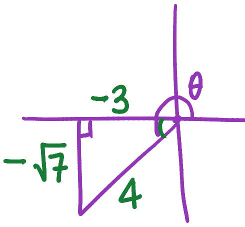
$\cos 135^\circ = -\frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$

$\cos 135^\circ \quad \cos 45^\circ$

Ex 3: Suppose θ is an angle in standard position with terminal arm in quadrant III and $\cos\theta = -\frac{3}{4}$. What are the exact values of $\sin\theta$ and $\tan\theta$? (Draw a diagram!)

← adj

← hyp



$(-3)^2+b^2=4^2$

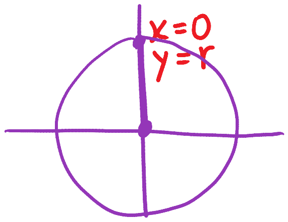
$b^2=7$

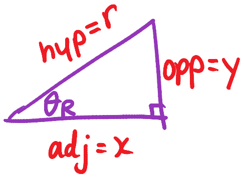
$b=\sqrt{7}$

$\sin\theta = -\frac{\sqrt{7}}{4}$

$\tan\theta = -\frac{\sqrt{7}}{-3} = \frac{\sqrt{7}}{3}$

Ex 4: If the terminal arm of a **quadrantal angle** θ is on the y-axis(i.e. $\theta = 90°$), determine:





a) $\sin\theta = \frac{opp}{hyp} = \frac{y}{r} = 1$

b) $\cos\theta = \frac{adj}{hyp} = \frac{x}{r} = 0$

c) $\tan\theta = \frac{opp}{adj} = \frac{y}{x} = \frac{y}{0}$ = undefined!

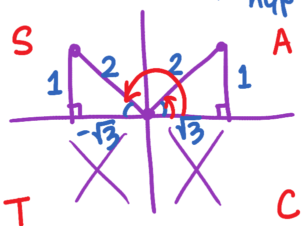
Solving for Angles given a trigonometric ratio:

Step 1: Determine which quadrants the solution(s) will be in. (Draw!)
Step 2: Find the reference angle. * Use special triangles or calculator
Step 3: Sketch the reference angle in the quadrants. Use the diagram to determine the standard position angle(s).

Ex 4: Solve for θ given:

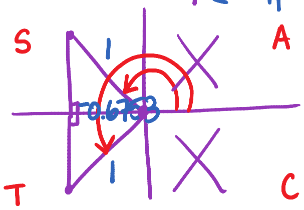
a) $\sin\theta = \frac{1 \leftarrow \text{opp}}{2 \leftarrow \text{hyp}}, 0^\circ \le \theta < 360^\circ$.

b) $\cos\theta = -\underset{1 \leftarrow \text{hyp}}{\overset{\swarrow \text{adj}}{\underline{0.6753}}}, 0^\circ \le \theta < 360^\circ$



Find θ_R:
→ Special Δ!
$\theta_R = 30^\circ$

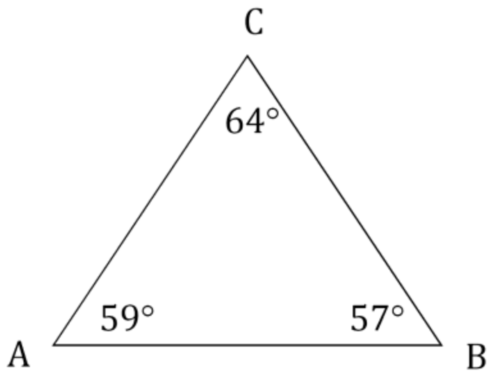
$$\boxed{\begin{aligned}\theta_1 &= 30^\circ\\ \theta_2 &= 150^\circ\end{aligned}}$$



Find θ_R:
→ Not special!
$\theta_R = \cos^{-1}(0.6753)$
$\theta_R = 47.5^\circ$

$$\boxed{\begin{aligned}\theta_1 &= 132.5^\circ\\ \theta_2 &= 227.5^\circ\end{aligned}}$$

2.3a: The Sine Law



Which of the following sides is the longest? Explain why.

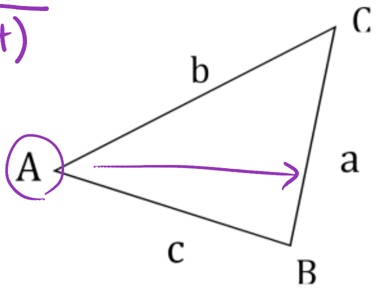
AB is the longest! Opposite the largest angle!
Relationship between angle & opposite side!

The Sine Law relates the sides to the opposite angles in **any triangle.**

(non-right)

$$\boxed{\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}}$$

*Reciprocal is also true!

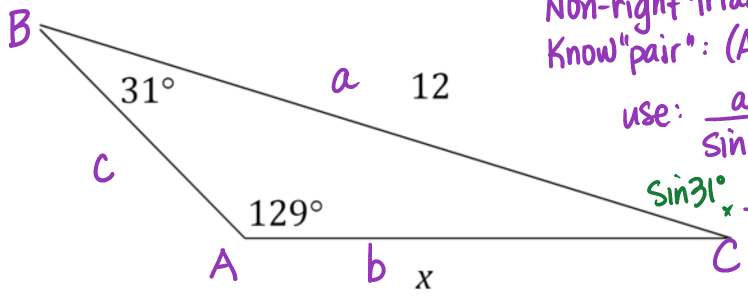


When do we use the sine law?

1) Non-right triangles.

2) A "pair": angle and opposite side (A and a)

Ex 1: Determine the side length *x* in the following diagram.



Non-right triangle!

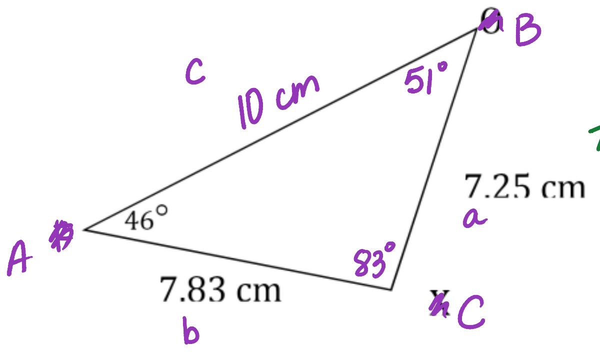
Know "pair": (A, a)

use: $\frac{a}{\sin A}=\frac{\textcircled{b}}{\sin B}$ ← unknown

$\sin 31° \times \frac{12}{\sin 129°}=\frac{x}{\cancel{\sin 31°}}\times \cancel{\sin 31°}$

$\boxed{x=7.95}$

Ex 2: Determine all of the missing sides and angles in the following triangle.



$\frac{\sin A}{a}=\frac{\sin \textcircled{B}}{b}$ ← unknown

$7.83 \times \frac{\sin 46°}{7.25}=\frac{\sin B}{\cancel{7.83}}\times \cancel{7.83}$

$0.7768=\sin B$

$B=\sin^{-1}(0.7768)$

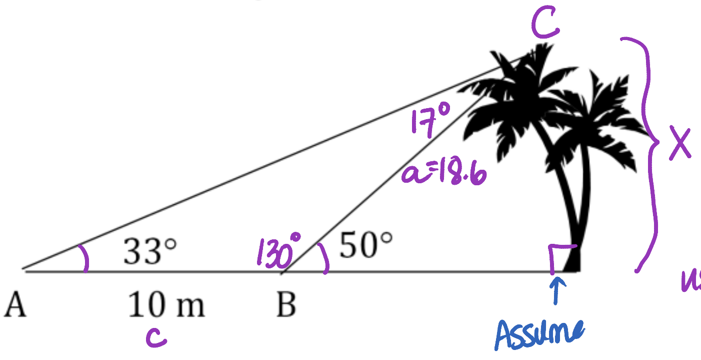
$B=51°$

$\frac{a}{\sin A}=\frac{\textcircled{c}}{\sin C}$ ← unknown

$\sin 83° \times \frac{7.25}{\sin 46°}=\frac{c}{\cancel{\sin 83°}}\times \cancel{\sin 83°}$

c = 10cm

Ex 3: Calculate the height of the tree if the distance from A to B is 10 metres.



$\frac{a}{\sin A}=\frac{c}{\sin C}$

$\frac{a}{\cancel{\sin 33°}}=\frac{10}{\sin 17°}\times \sin 33°$

$a=18.6$

use SOH CAH TOA:

$\sin 50°=\frac{X}{18.6}$

$\boxed{X=14.3\text{m}}$

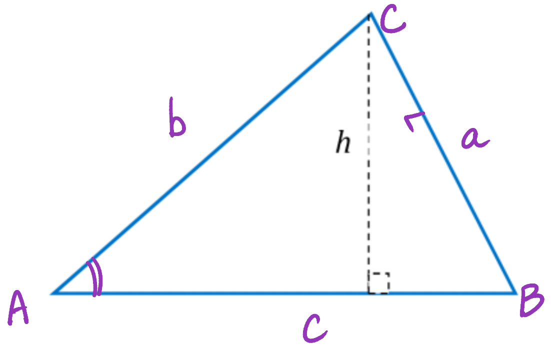
Why does the sine law work?

$b \times \sin A = \frac{h}{\cancel{b}} \times \cancel{b}$ $a \times \sin B = \frac{h}{\cancel{a}} \times \cancel{a}$

$h = \boxed{b\sin A} \overset{=}{\longleftrightarrow} h = \boxed{a\sin B}$

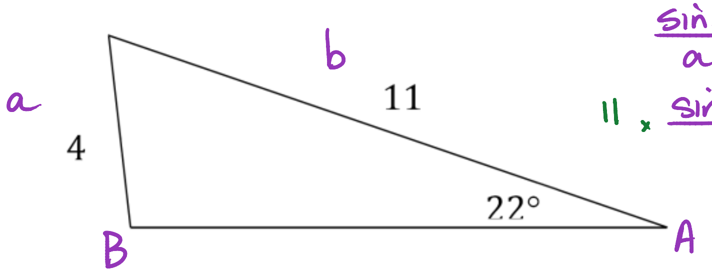
$\frac{\cancel{b}\sin A}{\cancel{b}a} = \frac{\cancel{a}\sin B}{\cancel{a}b}$

$\frac{\sin A}{a} = \frac{\sin B}{b}$



2.3b: The Ambiguous Case

Ex.1: Find the unknown sides and angles



- Know a 'pair' → Sine Law

$\frac{\sin A}{a} = \frac{\sin B}{b}$

$11 \times \frac{\sin 22^\circ}{4} = \frac{\sin B}{11} \times 11$

$\sin B = 1.03$

$B = \sin^{-1}(1.03)$

$= \text{error} ?!$

- Sometimes we cannot construct a triangle given certain information, or can construct more than one! → Since $a = 4$ is not long enough!

Number of Possible Triangles

Depending on the information given, we can limit the number of possible triangles. We must only check these cases when provided an Angle and the next two Sides (A.S.S.)

$< 90^\circ$

Given an acute angle:

I) Given $\angle A = 70^\circ, b = 10, a = 4$. The opposite side of $\angle A$ is too short it will not make a triangle

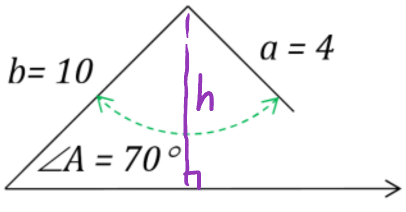
Find h → minimum length of a

$\sin A = \frac{\text{opp}}{\text{hyp}}$

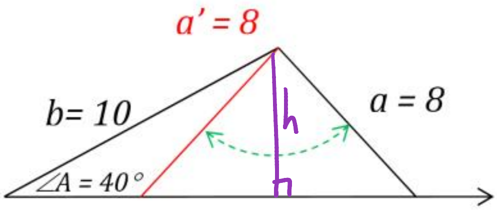
$\sin 70^\circ = \frac{h}{10}$

$h = 9.4$

since a is only 4, no possible Δ



II) Given $\angle A = 40°, b = 10, a = 8$. The opposite side of 40° is long enough to occupy 2 positions (known as the **ambiguous case**)



Find h:

$\sin 40° = \frac{h}{10}$ $h = 6.4$

Since $a = 8 > h = 6.4$ but less than $b = 10$

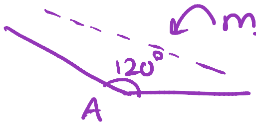
→ 2 Δ's possible!

Given an obtuse angle:

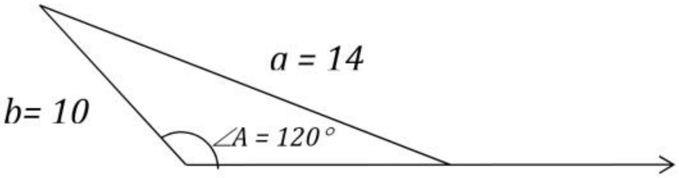
>90°

III) Given $\angle A = 120°, b = 14, a = 10$. The opposite side of 120° is not large enough to reach → no triangle possible.

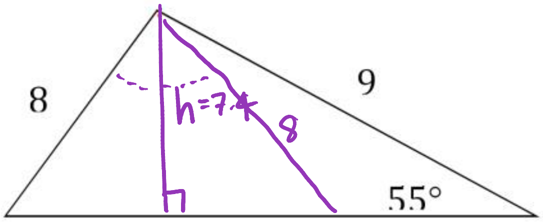
must be longest



IV) Given $\angle A = 120°, b = 10, a = 14$. The side opposite of 120° is large enough to reach → 1 triangle possible.



Ex.2: How many triangles are possible with the dimensions given?



Acute Angle, next 2 Sides (A.S.S)!

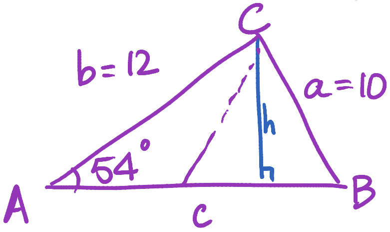
Find h: $\sin 55° = \frac{h}{9}$

$h = 7.4$

Since $a = 8 > h = 7.4$

→ 2 possible triangles!

Ex.3: Given ∠A = 54°, side b = 12, side a = 10. Solve the triangle.

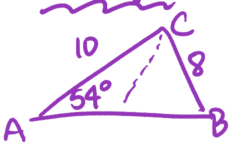


Acute angle, next 2 sides (A.S.S.!)

Find h: $\sin 54° = \frac{h}{12}$ $h=9.7$

Since $a=10 > h=9.7 \rightarrow$ 2 Triangles!

Triangle 1:



$\frac{\sin B}{12} = \frac{\sin 54°}{10}$

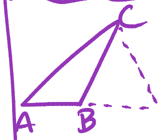
$\sin B = 0.97$

$\boxed{B = 76° \\ C = 50°}$

$\frac{c}{\sin 50°} = \frac{8}{\sin 54°}$

$\boxed{c = 7.6}$

Triangle 2:



B = obtuse version of Δ1 ($\theta_R = 76°$)

$\boxed{B = 104° \\ C = 22°}$

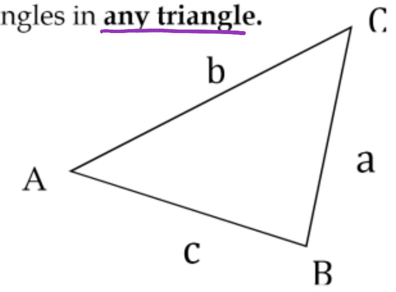
$\frac{c}{\sin 22°} = \frac{8}{\sin 54°}$

$\boxed{c = 3.7}$

2.4: The Cosine Law

The Cosine Law relates the sides to one of the opposite angles in **any triangle.** *non-right!

$$\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$$



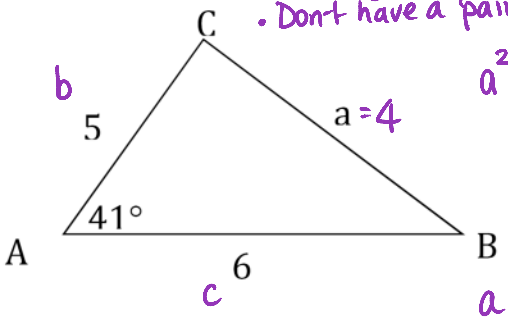
When do we use the cosine law?

- when we don't have a "pair"
- Given S.S.S. or SAS

Ex.1: Find side a

- Non-right Δ
- Don't have a 'pair'

} Cosine law!



$a^2 = b^2 + c^2 - 2bc\cos A$

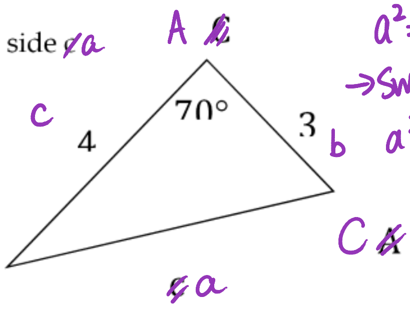
$= 5^2 + 6^2 - 2(5)(6)\cos 41°$

$= 15.717$

$a = \sqrt{15.717}$

$a = 3.96$ or 4.0

Ex.2: Find side ~~c~~ a



$a^2=b^2+c^2-2bc\cos A$

→Swap a's and c's

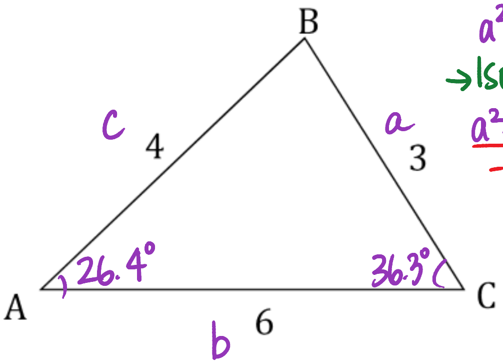
$a^2=3^2+4^2-2(3)(4)\cos 70°$

$=16.79$

$a=4.1$

→ $\boxed{c=4.1}$

Ex.3: Calculate ∠A and ∠C.



$-b^2-c^2$

$a^2=b^2+c^2-2bc\cos A$

→Isolate cosA

$\frac{a^2-b^2-c^2}{-2bc}=\frac{-\cancel{2bc}\cos A}{-\cancel{2bc}}$

$\cos A=\frac{a^2-b^2-c^2}{-2bc}$

$\cos A=\frac{(3^2-6^2-4^2)}{(-2(6)(4))}$

$\cos A=0.8958$

$\boxed{A=26.4°}$

Sine law for ∠C:

$\frac{\sin A}{a}=\frac{\sin C}{c}$

$4\times\frac{\sin 26.4°}{3}=\frac{\sin C}{\cancel{4}}\times\cancel{4}$

$0.5925=\sin C$

$\boxed{C=36.3°}$

Note: When solving triangles *non-right

SAS → Cosine Law
SSS → Cosine Law
ASS → Sine Law* → Ambiguous?
2 angles, one side → Sine Law

Ex.4: Solve for all sides and angles