Notes

February-16-16
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7.1: Absolute Value

For a real number *a*, the **absolute value** is always the non-negative value of the number. We show absolute value with two vertical lines, like brackets.

-7 0

Ex. 1: $|7| = 7$ $|-7| = 7$ $-|7| = -7$ $|0| = 0$

**Note: Think of it as a distance (from 0).

Ex. 2: Write the following real numbers in order from least to greatest.

6 5 4 1 3 2

$|-6.5|$, 5, $|4.75|$, -3.4, $\left|-\frac{12}{5}\right|$, $|-0.1|$

$= 6.5$ 5 4.75 -3.4 $\frac{12}{5}$ 0.1

In Calc: MATH → NUM: 1: abs( )

We treat absolute value symbols just like brackets. Use the order of operations.

Ex. 3: Evaluate $4 - |3(2) - 1| + 3$

$= 4 - |5| + 3$

$= 4 - 5 + 3 = 2$

Your Turn

Evaluate the following:

(a) $|4| - |-6| = 4 - 6 = \boxed{-2}$

(b) $5 - 3|2 - 7| = 5 - 3|-5| = 5 - 3(5) = \boxed{-10}$

(c) $|-2(5-7)^2 + 6| = |-2(-2)^2 + 6| = |-2(4) + 6| = |-2| = \boxed{2}$

Ex. 4: On stock markets, individual stock and bond values fluctuate a great deal, especially when the markets are volatile. A particular stock on the Toronto Stock Exchange (TSX) opened the month at \$13.55 per share, dropped to \$12.70, increased to \$14.05, and closed the month at \$13.85. Determine the total change in the value of this stock for the month.

Change 1: $|13.55-12.70| = 0.85$

Change 2: $|12.70-14.05| = 1.35$

Change 3: $|14.05-13.85| = 0.20$

Total = \$2.40

If there is a *variable* inside of our absolute value signs we no longer know whether or not to change the sign (i.e. is it positive or negative?). We must consider both cases – this can be summarized with **piecewise notation**.

In general: For any real number a: $|a| \begin{cases} a, & a \geq 0 \quad \text{(Case +)} \\ -a, & a < 0 \quad \text{(Case −)} \end{cases}$

change of sign

Ex. 5: Determine the **piecewise** notation for each of the following expressions:

(a) $|x-2| = \boxed{\begin{cases} x-2, & x \geq 2 \quad \text{(Case+)} \\ -x+2, & x<2 \quad \text{(Case−)} \end{cases}}$

(b) $|2x+1| = \boxed{\begin{cases} 2x+1, & x \geq -\frac{1}{2} \quad \text{(Case+)} \\ -2x-1, & x<-\frac{1}{2} \quad \text{(Case −)} \end{cases}}$

Case +: $x-2 \geq 0$

$x \geq 2$

Case +: $2x+1 \geq 0$

$2x \geq -1$

$x \geq -\frac{1}{2}$

7.2: Absolute Value Functions

$y=mx+b$

Ex. 1: Graph the function $y = x$ and its absolute value $y = |x|$ using a table of values. State the domain and range of both graphs.

$y=x$

x	y
-1	-1
0	0
1	1
2	2

D: $x \in \mathbb{R}$

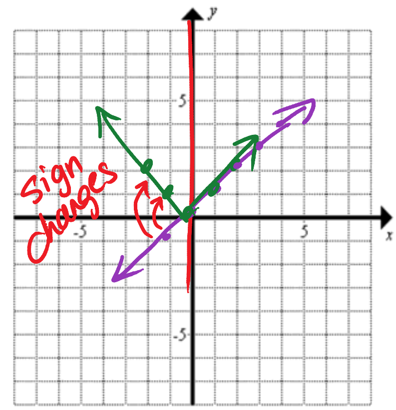
R: $y \in \mathbb{R}$

$y=|x|$

x	y
-2	2
-1	1
0	0
1	1
2	2
3	3

D: $x \in \mathbb{R}$

R: $y \geq 0$



* Check your graph with a calculator (TI-83 MATH->NUM->abs)

- What is the piecewise notation for the above graph $y = |x|$?

$$y = \begin{cases} x, & x \geq 0 \quad \text{(Case +)} \\ -x, & x < 0 \quad \text{(Case −)} \end{cases}$$

Ex. 2: Consider the absolute value function $y = |2x - 3|$

(a) Sketch the graph by comparing to y = 2x - 3
(b) Determine the x and y intercepts.
(c) Express the graph with piecewise notation.

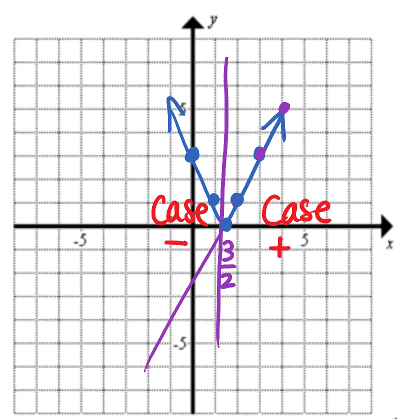
c) $$y = \begin{cases} 2x-3, & x \geq 3/2 \\ -2x+3, & x < 3/2 \end{cases}$$

Graph $y=2x-3$
$m=2$
$b=-3$

Graph $y=|2x-3|$
* change − to +
leave + the same (y)

b) x-int: $(y=0)$
$0=2x-3$
$\frac{3}{2}=x$

y-int: $(x=0)$
$y=|2(0)-3|$
$y=3$



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not varying

An **invariant point** is any point that remains unchanged when a transformation is applied.

- Can you name some invariant points in the above example?

$(\frac{3}{2}, 0), (2,1), (3,3), \ldots$

anything in Case + $(x \geq \frac{3}{2})$

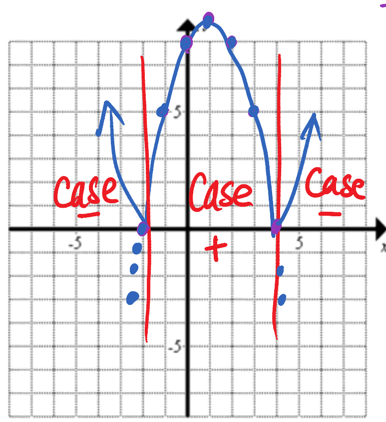
Ex. 3: Consider the absolute value function $y = |-(x-1)^2 + 9|$

(a) Sketch the graph.
(b) State the domain and range of the graph
(c). Determine the x and y intercepts.
(d) Express the graph with piecewise notation.

a) Sketch $y = -(x-1)^2 + 9$ $[y = a(x-p)^2 + q]$

vertex: $(1, 9)$
stretch: -1

over	down
1	1
2	4
3	9



b) Domain: $x \in \mathbb{R}$
Range: $y \geq 0$

c) x-int $(y=0)$

$0 = -(x-1)^2 + 9$
$-9 = -(x-1)^2$
$\pm\sqrt{9} = \sqrt{(x-1)^2}$
$\pm 3 = x - 1$
$x = \pm 3 + 1$
$= 4, -2$

y-int $(x=0)$

$y = |-(0-1)^2 + 9|$
$= |8|$
$= 8$

d) $y = \begin{cases} -(x-1)^2 + 9, & -2 \leq x \leq 4 \quad (+) \\ -(-(x-1)^2 + 9), & x < -2, x > 4 \quad (-) \end{cases}$

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7.3: Absolute Value Equations (day 1)

When solving an absolute value equation we must now consider two possibilities:

1. The value inside the absolute value is positive. → Case + → stays the same!
2. The value inside the absolute value is negative. → Case − → change the sign!

Consider $|x| = 10$. What is a possible solution? What is another possible solution?

$x = -10$ (Case −) or $x = 10$ (Case +)

Ex. 1: Solve $|x-3| = 4$ both algebraically and graphically.

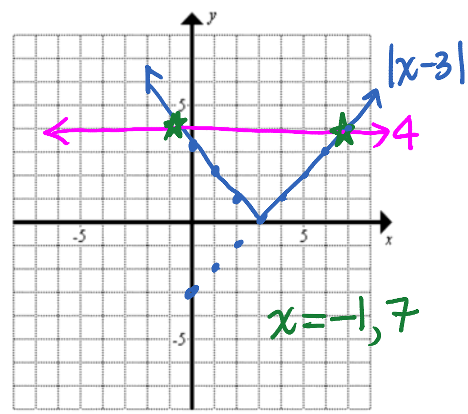
Case +: $x - 3 = 4$

$\boxed{x = 7}$ ✓

Case −: $-x + 3 = 4$

$-x = 1$

$\boxed{x = -1}$ ✓

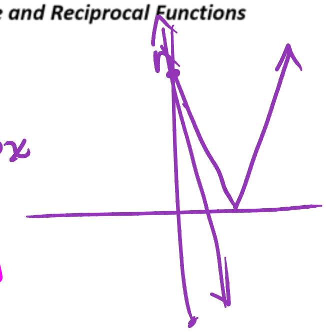


$\underbrace{|x-3|}_{y_1=} = \underbrace{4}_{y_2=}$

Solving absolute value equations:

1. Consider the positive and negative case for each absolute value:
2. Solve each case.
3. Check solution(s) by substituting the solution back into the ORIGINAL equation. Reject any that do not work (extraneous roots!).

Ex. 2: Solve $|2x-5| = 5-3x$



Case +: $2x-5=5-3x$

$5x-5=5$

$5x=10$

$x=2$ ← extraneous!

Case −: $-2x+5=5-3x$

$x+5=5$

$\boxed{x=0}$

Check: $|2(2)-5| = 5-3(2)$

$|-1| = -1$ ✗

$|2(0)-5| = 5-3(0)$

$|-5| = 5$ ✓

YOUR TURN

Solve $|x+5| = 4x-1$

case +: $x+5=4x-1$

$6=3x$

$\boxed{2=x}$

Case −: $-x-5=4x-1$

$-4=5x$

$-\frac{4}{5}=x$ ← extraneous

Check: $|2+5| = 4(2)-1$ ✓

$\left|-\frac{4}{5}+5\right| = 4\left(-\frac{4}{5}\right)-1$ ✗

Ex. 3: Solve $|3x-4|+12=9$

$|3x-4|=-3$ (No Solution!)

$+ \neq -$

Case +: $3x-4+12=9$

$3x+8=9$

$3x=1$

$x=\frac{1}{3}$ ← extraneous →

Case −: $-3x+4+12=9$

$-3x+16=9$

$-3x=-7$

$x=\frac{7}{3}$

Check: $\left|\cancel{3}\left(\frac{1}{\cancel{3}}\right)-4\right|^{15}+12=9$ ✗

$\left|\cancel{3}\left(\frac{7}{\cancel{3}}\right)-4\right|^{15}+12=9$ ✗

$\boxed{\text{No Solution}}$

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7.3: Absolute Value Equations (day 2)

Sometimes we may have more complex absolute value equations to solve. They may have quadratic equations, or more than 1 absolute value function. The general steps will be the same as before.

Ex. 4: Solve $|x-10| = x^2 - 10x$

Case +: $x-10 = x^2 - 10x$

$0 = x^2 - 11x + 10$

$0 = (x-1)(x-10)$

$x = 1, 10$

Case −: $-x+10 = x^2 - 10x$

$0 = x^2 - 9x - 10$

$0 = (x-10)(x+1)$

$x = 10, -1$

Check:

$x = 1$ ✗

$\boxed{x = 10 \ ✓ \quad x = -1 \ ✓}$

Ex. 5: Solve $|x^2 - 2x| = 1$

Case +: $x^2 - 2x = 1$

Doesn't Factor... ☹ → $x^2 - 2x - 1 = 0$

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2(1)}$

$= \frac{2 \pm \sqrt{8}}{2}$

$= \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

$\approx 2.41, -0.41$

Check

Case −: $-x^2 + 2x = 1$

$0 = x^2 - 2x + 1$

$0 = (x-1)(x-1)$

$x = 1$ check

$\boxed{x = 1 \pm \sqrt{2}, 1}$

Ex. 6: Solve $|x+1| + |x+2| = 5$

Case ++: $x+1+x+2 = 5$

$2x + 3 = 5$

$2x = 2$

$x = 1$

Case +−: $\cancel{x} + 1 - \cancel{x} - 2 = 5$

$-1 = 5$

N/A

Case −+: $-\cancel{x} - 1 + \cancel{x} + 2 = 5$

$1 = 5$

N/A

Case −−: $-x - 1 - x - 2 = 5$

$-2x - 3 = 5$

$-2x = 8$

$x = -4$

Check: $\boxed{x = 1, -4}$

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7.4: Reciprocal Functions (day 1)

Given a function $y = f(x)$, its corresponding **reciprocal function** is $y = \frac{1}{f(x)}$

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Ex.1: Determine the reciprocal of each function. Are there non-permissible values?

(a) $y = 2x + 3$

$y = \frac{1}{2x+3}$ npv: $x \neq -3/2$

$2x+3=0$

$x = -\frac{3}{2}$

(b) $y = \frac{3}{x-1}$ $x \neq 1$

$y = \frac{x-1}{3}$ no npv's

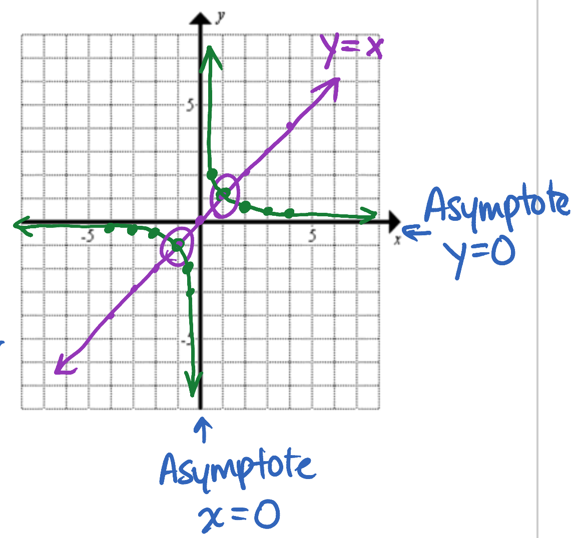
Ex.2: Sketch the graph of $y = x$ and its reciprocal function using a table of values.

line

$y = \frac{1}{x}$

x	y
−4	−1/4
−3	−1/3
−2	−1/2
−1	−1
0	error (npv)
1	1
2	1/2
3	1/3
4	1/4

Bigger Smaller



1/2 2

A few questions about example 2)

- Why does the curve approach the y-axis but never touch it?

  → $x=0$: $y = \frac{1}{0}$ is undefined!

- Why does the curve approach the x-axis but never touch it?

  → $y=0$: $0 = \frac{1}{x}$ has no solution.

- Recall that **invariant points** are those that are unchanged. What are the invariant points for this pair of functions? What is special about the reciprocals of these values?

  (1, 1) and (−1, −1)

  Their reciprocals are the same!

Asymptote: A line whose distance from a curve approaches zero. (More on asymptotes next class!)

Ex.3: Complete the following table:

Characteristic	$y = x$	$y = \frac{1}{x}$
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}, x \neq 0$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}, y \neq 0$
Invariant points	$(1,1)$	$(-1,-1)$

7.4: Reciprocal Functions (day 2)

Ex.1: Consider $y = x - 3$

(a) Determine its reciprocal function $y = \frac{1}{f(x)}$

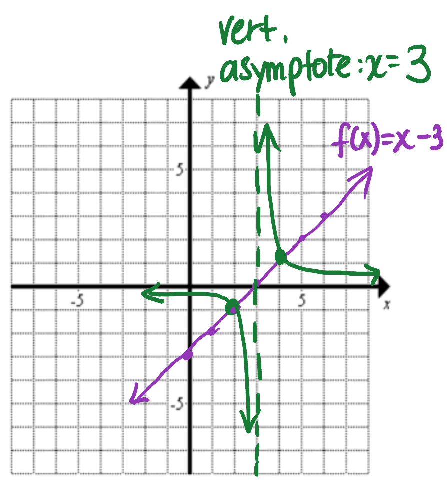
(b) Determine the equation of the vertical asymptote of reciprocal function.

(c) Graph both $f(x)$ and $\frac{1}{f(x)}$.

a) Reciprocal: $y = \frac{1}{x-3}$ n.p.v. $x \neq 3$

b) vert. Asymptote: $x = 3$

c) $f(x) = x - 3$ Invariant Points when $y = \pm 1$



* Check on your graphing calculator

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Graphing reciprocal functions (without a calculator!)

(zeros)

1. Graph the function $f(x)$. Mark the x-intercept(s) and points where $f(x) = \pm 1$ (**invariant points**).
2. Mark the **vertical asymptotes** of the reciprocal function at the x-intercepts.
3. Create the graph through the invariant points tending towards the vertical asymptotes and the x-axis. Check with a table of values or graphing calculator if needed.

Ex.2: Consider $f(x) = x^2 - 4$

(a) What is the reciprocal function of $f(x)$?

(b) State the non-permissible values of x and the equation(s) of the vertical asymptote(s) of the reciprocal function.

(c) What are the invariant points where $f(x) = \pm 1$?

(d) Graph the function and its reciprocal.

a) Reciprocal: $y = \frac{1}{x^2-4} = \frac{1}{(x+2)(x-2)}$

b) n.p.v.: $x \neq \pm 2$

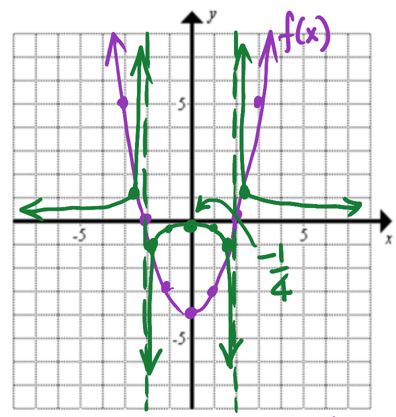
vert. asymptotes: $x = 2$ and $x = -2$

c) $y = \pm 1 \rightarrow$ plug in to $f(x)$ or $\frac{1}{f(x)}$

$1 = x^2 - 4$ and $-1 = x^2 - 4$

$5 = x^2$ $3 = x^2$

$\pm\sqrt{5} = x$ $(\pm\sqrt{5}, 1)$ $\pm\sqrt{3} = x$ $(\pm\sqrt{3}, -1)$



$\frac{1}{f(x)}$: Domain: $x \in \mathbb{R}, x \neq \pm 2$

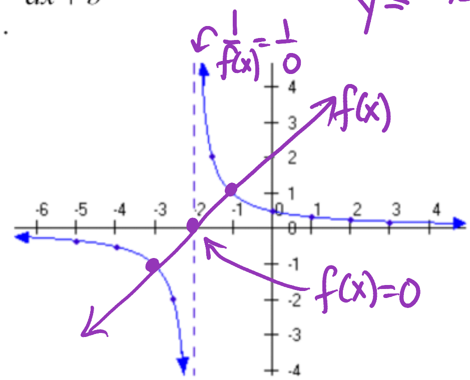
Range: $y > 0$

$y \leq -1/4$

Ex.3: Given the graph of a reciprocal function of the form $y = \dfrac{1}{ax+b}$:

(a) Sketch the graph of the original function $y = f(x)$.

(b) Determine the original function $y = f(x)$



b) linear function

slope = 1

y-int = 2

$\boxed{f(x) = x + 2}$