1 Fundamentals

Introduction

Mathematics is an exciting field of study, concerned with structure, patterns and ideas. To fully appreciate and understand these core aspects of mathematics, you need to be confident and skilled in the rules and language of algebra. Although you have encountered some, perhaps most or all, of the material in this chapter in a previous mathematics course, the aim of this chapter is to ensure that you are familiar with fundamental terminology, notation and algebraic techniques.

1.1 The real numbers

The most fundamental building blocks in mathematics are numbers and the operations that can be performed on them. Algebra, like arithmetic, involves performing operations such as addition, subtraction, multiplication and division on numbers. In arithmetic, we are performing operations on known, specific, numbers (e.g. $5 + 3 = 8$). However, in algebra we often deal with operations on unknown numbers represented by variables – usually symbolized by a letter $\left(\text{e.g. } \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}\right)$. The use of variables gives us the power to write general statements indicating relationships between numbers. But what types of numbers can variables represent? All of the mathematics in this course involves the real numbers and subsets of the real numbers.

> The word *algebra* comes from the 9th-century Arabic book *Hisâb al-Jabr w'al-Muqabala*, written by al-Khowarizmi. The title refers to transposing and combining terms, two processes used in solving equations. In Latin translations, the title was shortened to *Aljabr*, from which we get the word *algebra*. The author's name made its way into the English language in the form of the word *algorithm*.

A real number is any number that can be represented by a point on the real number line (Figure 1.1). Each point on the real number line corresponds to one and only one real number, and each real number corresponds to one and only one point on the real number line. This kind of relationship is called a **one-to-one correspondence**. The number associated with a point on the real number line is called the **coordinate** of the point.

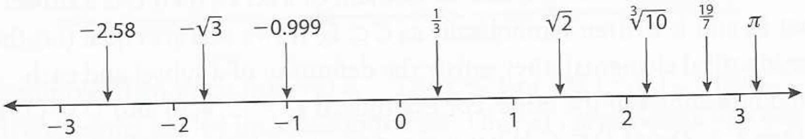


Figure 1.1 The real number line.

Subsets of the real numbers

The set of real numbers $\mathbb{R}$ contains some important subsets with which you should be familiar.

When we first learn to count, we use the numbers 1, 2, 3, … . These numbers form the set of *counting* numbers or **positive integers** $\mathbb{Z}^+$.

• **Hint:** Do not be confused if you see other textbooks indicate that the set $\mathbb{N}$ (usually referred to as the natural numbers) does *not* include zero – and is defined as $\mathbb{N} = \{1, 2, 3, \ldots\}$. There is disagreement among mathematicians whether zero should be considered a natural number – i.e. reflecting how we *naturally* count. We normally do not start counting at zero. However, zero does represent a counting concept in that it is the absence of any objects in a set. Therefore, some mathematicians (and the IB mathematics curriculum) define the set $\mathbb{N}$ as the positive integers *and* zero.

Adding zero to the positive integers (0, 1, 2, 3, ...) forms the set referred to as the set $\mathbb{N}$ in IB notation.

The set of **integers** consists of the counting numbers with their corresponding negative values and zero $(\ldots -3, -2, -1, 0, 1, 2, 3, \ldots)$ and is denoted by $\mathbb{Z}$ (from the German word *Zahl* for number).

We construct the **rational numbers** $\mathbb{Q}$ by taking ratios of integers. Thus, a real number is rational if it can be written as the ratio $\frac{p}{q}$ of any two integers, where $q \neq 0$. The decimal representation of a rational number either repeats or terminates. For example, $\frac{5}{7} = 0.714\,285\,714\,285\ldots = 0.\overline{714\,285}$ (the block of six digits repeats) or $\frac{3}{8} = 0.375$ (the decimal terminates at 5, or, alternatively, has a repeating zero after the 5).

A real number that cannot be written as the ratio of two integers, such as π and $\sqrt{2}$, is called **irrational**. Irrational numbers have infinite non-repeating decimal representations. For example, $\sqrt{2} \approx 1.414\,213\,5623\ldots$ and $\pi \approx 3.141\,592\,653\,59\ldots$. There is no special symbol for the set of irrational numbers.

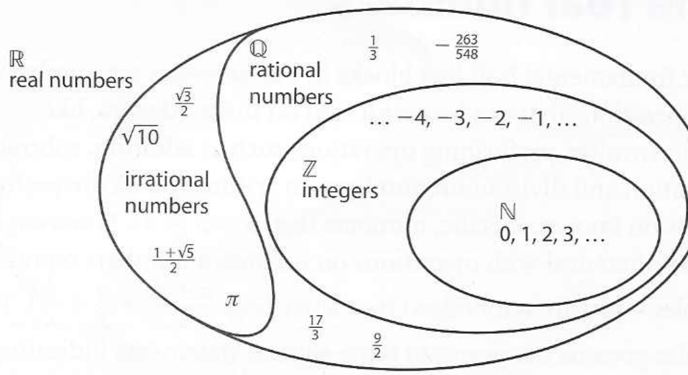


Figure 1.2 A Venn diagram representing the relationships between the different subsets of the real numbers. The rational numbers combined with the irrational numbers make up the entire set of real numbers.

Table 1.1 A summary of the subsets of the real numbers $\mathbb{R}$ and their symbols.

Positive integers	$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$
Positive integers and zero	$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$
Integers	$\mathbb{Z} = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$
Rational numbers	$\mathbb{Q}$ = any number that can be written as the ratio $\frac{p}{q}$ of any two integers, where $q \neq 0$

Sets and intervals

If every element of a set C is also an element of a set D, then C is a **subset** of set D, and is written symbolically as $C \subseteq D$. If two sets are equal (i.e. they have identical elements), they satisfy the definition of a subset and each would be a subset of the other. For example, if $C = \{2, 4, 6\}$ and $D = \{2, 4, 6\}$, then $C \subseteq D$ and $D \subseteq C$. What is more common is that a subset is a set that is contained in a larger set and does not contain at least one element of the larger set. Such a subset is called a **proper subset** and is denoted with the symbol $\subset$. For example, if $D = \{2, 4, 6\}$ and $E = \{2, 4\}$, then E is a proper subset of D and is written $E \subset D$. All of the subsets of the real numbers discussed earlier in this section are proper subsets of the real numbers, for example, $\mathbb{N} \subset \mathbb{R}$ and $\mathbb{Z} \subset \mathbb{R}$.

The symbol $\in$ indicates that a number, or a number assigned to a variable, belongs to (is an element of) a set. We can write $6 \in \mathbb{Z}$, which is read '6 is an element of the set of integers'. Some sets can be described by listing their elements within brackets. For example, the set A that contains all of the integers between -2 and 2 inclusive can be written as $A = \{-2, -1, 0, 1, 2\}$. We can also use set-builder notation to indicate that the elements of set A are the values that can be assigned to a particular variable. For example, the notation $A = \{x \mid -2, -1, 0, 1, 2\}$ or $A = \{x \in \mathbb{Z} \mid -2 \leqslant x \leqslant 2\}$ indicates that 'A is the set of all x such that x is an integer greater than or equal to -2 and less than or equal to positive 2'. Set-builder notation is particularly useful for representing sets for which it would be difficult or impossible to list all of the elements. For example, to indicate the set of positive integers n greater than 5, we could write $\{n \in \mathbb{Z} \mid n > 5\}$ or $\{n \mid n > 5, n \in \mathbb{Z}\}$.

The **intersection** of A and B (Figure 1.3), denoted by $A \cap B$ and read 'A intersection B', is the set of all elements that are in both set A *and* set B. The **union** of two sets A and B (Figure 1.4), denoted by $A \cup B$ and read 'A union B', is the set of all elements that are in set A *or* in set B (or in both). The set that contains no elements is called the **empty set** (or null set) and is denoted by $\varnothing$.

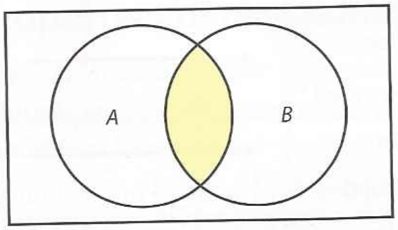


Figure 1.3 Intersection of sets A and B. $A \cap B$

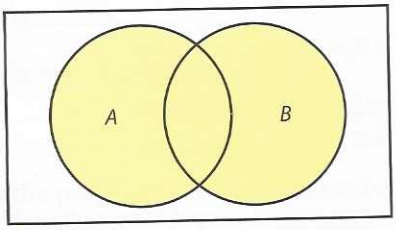
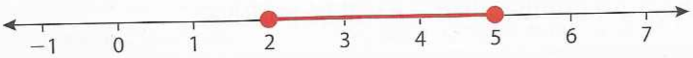


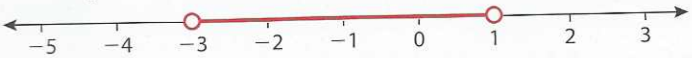
Figure 1.4 Union of sets A and B. $A \cup B$

• **Hint:** Unless indicated otherwise, if interval notation is used, we assume that it indicates a subset of the real numbers. For example, the expression $x \in [-3, 3]$ is read 'x is any real number between -3 and 3 inclusive.'

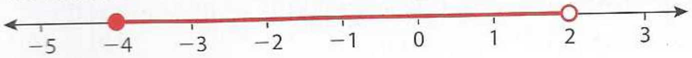
Some subsets of the real numbers are a portion, or an **interval**, of the real number line and correspond geometrically to a line segment or a ray. They can be represented either by an inequality or by interval notation. For example, the set of all real numbers x between 2 and 5, including 2 and 5, can be expressed by the inequality $2 \leqslant x \leqslant 5$ or by the interval notation $x \in [2, 5]$. This is an example of a **closed interval** (i.e. both endpoints are included in the set) and corresponds to the line segment with endpoints of $x = 2$ and $x = 5$.



An example of an **open interval** is $-3 < x < 1$ or $x \in]-3, 1[$, where both endpoints are *not* included in the set. This set corresponds to a line segment with 'open dots' on the endpoints indicating they are excluded.



If an interval, such as $-4 \leqslant x < 2$ or $x \in [-4, 2[$, includes one endpoint but not the other, it is referred to as a **half-open** interval.



● **Hint:** The symbols ∞ (positive infinity) and −∞ (negative infinity) do not represent real numbers. They are simply symbols used to indicate that an interval extends indefinitely in the positive or negative direction.

The three examples of intervals on the real number line given above are all considered **bounded** intervals in that they are line segments with two endpoints (regardless of whether included or excluded). The set of all real numbers greater than 2 is an open interval because the one endpoint is excluded and can be expressed by the inequality $x > 2$, or $x \in (2, \infty)$. This is also an example of an **unbounded** interval and corresponds to a portion of the real number line that is a ray.

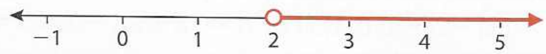
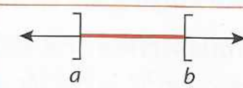
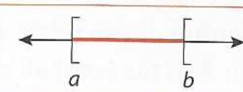
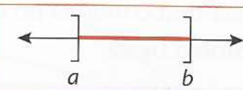
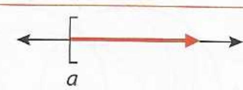
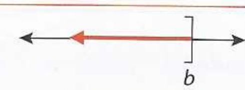
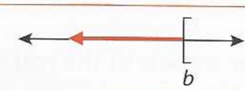


Table 1.2 The nine possible types of intervals – both bounded and unbounded. For all of the examples given, we assume that $a < b$.

Interval notation	Inequality	Interval type	Graph
$x \in [a, b]$	$a \leqslant x \leqslant b$	closed bounded	
$x \in]a, b[$	$a < x < b$	open bounded	
$x \in [a, b[$	$a \leqslant x < b$	half-open bounded	
$x \in]a, b]$	$a < x \leqslant b$	half-open bounded	
$x \in [a, \infty[$	$x \geqslant a$	half-open unbounded	
$x \in]a, \infty[$	$x > a$	open unbounded	
$x \in]-\infty, b]$	$x \leqslant b$	half-open unbounded	
$x \in]-\infty, b[$	$x < b$	open unbounded	
$x \in]-\infty, \infty[$	real number line		

Absolute value (modulus)

The **absolute value** (or modulus) of a number a, denoted by $|a|$, is the distance from a to 0 on the real number line. Since a distance must be positive or zero, the absolute value of a number is never negative. Note that if a is a negative number then $-a$ will be positive.

Definition of absolute value

If a is a real number, the **absolute value** of a is

$$|a| = \begin{cases} a & \text{if } a \geqslant 0 \\ -a & \text{if } a < 0 \end{cases}$$

Here are four useful properties of absolute value:

Given that a and b are real numbers, then:

1. $|a| \geqslant 0$ **2.** $|-a| = |a|$ **3.** $|ab| = |a||b|$ **4.** $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, b \neq 0$

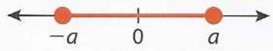
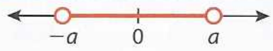
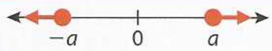
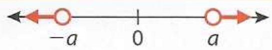
Absolute value is used to define the distance between two numbers on the real number line.

Distance between two points on the real number line

Given that a and b are real numbers, the distance between the points with coordinates a and b on the real number line is $|b-a|$, which is equivalent to $|a-b|$.

Absolute value expressions can appear in inequalities, as shown in the table below.

Table 1.3 Properties of absolute value inequalities.

Inequality	Equivalent form	Graph
$\|x\| \leqslant a$	$-a \leqslant x \leqslant a$	
$\|x\| < a$	$-a < x < a$	
$\|x\| \geqslant a$	$x \leqslant -a$ or $x \geqslant a$	
$\|x\| > a$	$x < -a$ or $x > a$	

Properties of real numbers

There are four arithmetic operations with real numbers: addition, multiplication, subtraction and division. Since subtraction can be written as addition $(a - b = a + (-b))$, and division can be written as multiplication $\left(\frac{a}{b} = a\left(\frac{1}{b}\right), b \neq 0\right)$, then the properties of the real numbers are defined in terms of addition and multiplication only. In these definitions, $-a$ is the **additive inverse** (or opposite) of a, and $\frac{1}{a}$ is the **multiplicative inverse** (or reciprocal) of a.

Table 1.4 Properties of real numbers.

Property	Rule	Example
commutative property of addition:	$a + b = b + a$	$2x^3 + y = y + 2x^3$
commutative property of multiplication:	$ab = ba$	$(x-2)3x^2 = 3x^2(x-2)$
associative property of addition:	$(a+b)+c = a+(b+c)$	$(1+x)-5x = 1+(x-5x)$
associative property of multiplication:	$(ab)c = a(bc)$	$(3x \cdot 5y)\left(\frac{1}{y}\right) = (3x)\left(5y \cdot \frac{1}{y}\right)$
distributive property:	$a(b+c) = ab + ac$	$x^2(x-2) = x^2 \cdot x + x^2(-2)$
additive identity property:	$a + 0 = a$	$4y + 0 = 4y$
multiplicative identity property:	$1 \cdot a = a$	$\frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{4}{4} \cdot \frac{2}{3} = \frac{8}{12}$
additive inverse property:	$a + (-a) = 0$	$6y^2 + (-6y^2) = 0$
multiplicative inverse property:	$a \cdot \frac{1}{a} = 1, a \neq 0$	$(y-3)\left(\frac{1}{y-3}\right) = 1$

Note: These properties can be applied in either direction as shown in the 'rules' above.

Exercise 1.1

In questions 1–6, plot the two real numbers on the real number line, and then find the distance between their coordinates.

1 $5; \frac{3}{4}$ **2** $-2; -11$ **3** $13.4; 6$

4 $7; -\frac{5}{3}$ **5** $-3\pi; \frac{2\pi}{3}$ **6** $-\frac{5}{6}; -\frac{9}{4}$

In questions 7–12, write an inequality to represent the given interval and state whether the interval is closed, open or half-open. Also, state whether the interval is bounded or unbounded.

7 $[-5, 3]$ **8** $]-10, -2]$ **9** $[1, \infty[$

10 $]-\infty, 4[$ **11** $[0, 2\pi[$ **12** $[a, b]$

In questions 13–18, use interval notation to represent the subset of real numbers that is indicated by the inequality.

13 $x > 6$ **14** $x \leqslant -8$ **15** $2 < x < 9$

16 $0 \leqslant x < 12$ **17** $x > -5$ **18** $-3 \leqslant x \leqslant 3$

In questions 19–22, use inequality and interval notation to represent the given subset of real numbers.

19 x is at least 6.

20 x is greater than or equal to 4 and less than 10.

21 x is negative.

22 x is any positive number less than 25.

In questions 23–28, state the indicated set given that $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{2, 4, 6\}$.

23 $A \cap B$ **24** $A \cup B$ **25** $B \cap C$

26 $A \cup C$ **27** $A \cap C$ **28** $A \cup B \cup C$

In questions 29–32, use the symbol $\subset$ to write a correct statement involving the two sets.

29 $\mathbb{Z}$ and $\mathbb{R}$ **30** $\mathbb{N}$ and $\mathbb{Q}$ **31** $\mathbb{Z}$ and $\mathbb{N}$ **32** $\mathbb{Q}$ and $\mathbb{Z}$

In questions 33–36, express the inequality, or inequalities, using absolute value.

33 $-6 < x < 6$ **34** $x \leqslant -4$ or $x \geqslant 4$

35 $-\pi \leqslant x \leqslant \pi$ **36** $x < -1$ or $x > 1$

In questions 37–42, evaluate each absolute value expression.

37 $|-13|$ **38** $|7-11|$ **39** $-5|-5|$

40 $|-3| - |-8|$ **41** $|\sqrt{3} - 3|$ **42** $\frac{-1}{|-1|}$

In questions 43–46, find all values of x that make the equation true.

43 $|x| = 5$ **44** $|x - 3| = 4$

45 $|6 - x| = 10$ **46** $|3x + 5| = 1$

1.2 Roots and radicals (surds)

Roots

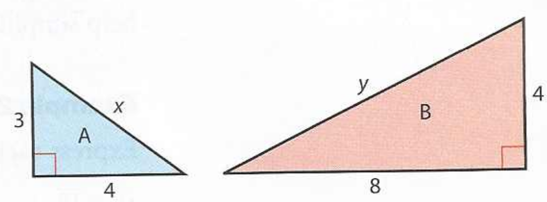
If a number can be expressed as the product of two equal factors, that factor is called the **square root** of the number. For example, 7 is the square root of 49 because $7 \times 7 = 49$. Now, 49 is also equal to -7×-7, so -7 is also a square root of 49. Every positive real number will have two real number square roots – one positive and one negative. However, there are many instances where we want only the positive square root. The symbol $\sqrt{\ }$ (sometimes called the root or radical symbol) indicates only the positive square root – often referred to as the **principal square root**. In words, the square roots of 16 are 4 and -4; but, symbolically, $\sqrt{16} = 4$. The negative square root of 16 is written as $-\sqrt{16}$, and when both square roots are wanted we write $\pm\sqrt{16}$.

When a number can be expressed as the product of three equal factors, then that factor is called the **cube root** of the number. For example, -4 is the cube root of -64 because $(-4)(-4)(-4) = -64$. This is written symbolically as $\sqrt[3]{-64} = -4$.

In general, if a number a can be expressed as the product of n equal factors then that factor is called the ***n*th root** of a and is written as $\sqrt[n]{a}$. n is called the **index** and if no index is written it is assumed to be a 2, thereby indicating a square root. If n is an even number (e.g. square root, fourth root, etc.) then the **principal *n*th root** is positive. For example, since $(-2)(-2)(-2)(-2) = 16$, then -2 is a fourth root of 16. However, the principal fourth root of 16, written $\sqrt[4]{16}$, is equal to $+2$.

Radicals (surds)

Some roots are rational and some are irrational. Consider the two right triangles in Figure 1.5. By applying Pythagoras' theorem, we find the length of the hypotenuse for triangle A to be exactly 5 (an integer and rational number) and the hypotenuse for triangle B to be exactly $\sqrt{80}$ (an irrational number). An irrational root – e.g. $\sqrt{80}, \sqrt{3}, \sqrt{10}, \sqrt[3]{4}$ – is called a **radical** or **surd**. The only way to express irrational roots exactly is in radical, or surd, form.



$x^2 = 3^2 + 4^2$
$x^2 = 9 + 16$
$x^2 = 25$
$\sqrt{x^2} = \sqrt{25}$
$x = \sqrt{25}$
$x = 5$

$y^2 = 4^2 + 8^2$
$y^2 = 16 + 64$
$y^2 = 80$
$\sqrt{y^2} = \sqrt{80}$
$y = \sqrt{80}$

Figure 1.5

It is not immediately obvious that the following expressions are all equivalent.

$$\sqrt{80},\ 2\sqrt{20},\ \frac{16\sqrt{5}}{\sqrt{16}},\ 2\sqrt{2}\sqrt{10},\ \frac{10\sqrt{8}}{\sqrt{10}},\ 4\sqrt{5},\ 5\sqrt{\frac{16}{5}}$$

Square roots occur frequently in several of the topics in this course, so it will be useful for us to be able to simplify radicals and recognise equivalent radicals. Two useful rules for manipulating expressions with radicals are given below.

• **Hint:** The solution for the hypotenuse of triangle A in Figure 1.5 involves the equation $x^2 = 25$. Because x represents a length that must be positive, we want only the positive square root when taking the square root of both sides of the equation – i.e. $\sqrt{25}$. However, if there were no constraints on the value of x, we must remember that a positive number will have two square roots and we would write $\sqrt{x^2} = \pm\sqrt{25} \Rightarrow x = \pm 5$.

Simplifying radicals

For $a \geqslant 0$, $b \geqslant 0$ and $n \in \mathbb{Z}^+$, the following rules can be applied:

1 $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ **2** $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Note: Each rule can be applied in either direction.

Example 1

Simplify each of the radicals.

a) $\sqrt{5} \times \sqrt{5}$ b) $\sqrt{2} \times \sqrt{18}$ c) $\dfrac{\sqrt{48}}{\sqrt{3}}$ d) $\sqrt[3]{6} \times \sqrt[3]{36}$

Solution

a) $\sqrt{5} \times \sqrt{5} = \sqrt{5 \cdot 5} = \sqrt{25} = 5$

Note: A special case of the rule $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ when $n = 2$ is $\sqrt{a} \times \sqrt{a} = a$.

b) $\sqrt{2} \times \sqrt{18} = \sqrt{2 \cdot 18} = \sqrt{36} = 6$

c) $\dfrac{\sqrt{48}}{\sqrt{3}} = \sqrt{\dfrac{48}{3}} = \sqrt{16} = 4$

d) $\sqrt[3]{6} \times \sqrt[3]{36} = \sqrt[3]{6 \cdot 36} = \sqrt[3]{216} = 6$

The radical $\sqrt{24}$ can be simplified because one of the factors of 24 is 4, and the square root of 4 is rational (i.e. 4 is a perfect square).

$$\sqrt{24} = \sqrt{4 \cdot 6} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$$

Rewriting 24 as the product of 3 and 8 (rather than 4 and 6) would not help simplify $\sqrt{24}$ because neither 3 nor 8 are perfect squares.

Example 2

Express each in terms of the simplest possible radical.

a) $\sqrt{18}$ b) $\sqrt{80}$ c) $\sqrt{\dfrac{3}{25}}$ d) $\sqrt{1000}$

Solution

a) $\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$

b) $\sqrt{80} = \sqrt{16 \cdot 5} = \sqrt{16}\sqrt{5} = 4\sqrt{5}$

Note: 4 is a factor of 80 and is a perfect square, but 16 is the largest factor that is a perfect square.

c) $\sqrt{\dfrac{3}{25}} = \dfrac{\sqrt{3}}{\sqrt{25}} = \dfrac{\sqrt{3}}{5}$

d) $\sqrt{1000} = \sqrt{100 \cdot 10} = \sqrt{100}\sqrt{10} = 10\sqrt{10}$

We prefer not to have radicals in the denominator of a fraction. Recall, from Example 1a), the special case of the rule $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$ when $n = 2$ is $\sqrt{a} \times \sqrt{a} = a$. The process of eliminating irrational numbers from the denominator is called **rationalising the denominator**.

Example 3

Rationalise the denominator of each expression. a) $\frac{2}{\sqrt{3}}$ b) $\frac{\sqrt{7}}{4\sqrt{10}}$

Solution

a) $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

b) $\frac{\sqrt{7}}{4\sqrt{10}} = \frac{\sqrt{7}}{4\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{\sqrt{70}}{4 \cdot 10} = \frac{\sqrt{70}}{40}$

Exercise 1.2

In questions 1–9, express each in terms of the simplest possible radical.

1 $\sqrt{8}$ **2** $\frac{\sqrt{28}}{\sqrt{7}}$ **3** $\sqrt{3} \times \sqrt{12}$

4 $\sqrt[3]{9} \times \sqrt[3]{3}$ **5** $\frac{\sqrt[4]{64}}{\sqrt[4]{4}}$ **6** $\sqrt{\frac{15}{20}}$

7 $\sqrt{50}$ **8** $\sqrt{63}$ **9** $\sqrt{288}$

In questions 10–13, completely simplify the expression.

10 $7\sqrt{2} - 3\sqrt{2}$ **11** $\sqrt{12} + 8\sqrt{3}$

12 $\sqrt{300} + 5\sqrt{2} - \sqrt{72}$ **13** $\sqrt{75} + 2\sqrt{24} - \sqrt{48}$

In questions 14–19, rationalise the denominator, simplifying if possible.

14 $\frac{1}{\sqrt{2}}$ **15** $\frac{3}{\sqrt{5}}$ **16** $\frac{2\sqrt{3}}{\sqrt{7}}$

17 $\frac{1}{\sqrt{27}}$ **18** $\frac{8}{3\sqrt{2}}$ **19** $\frac{\sqrt{12}}{\sqrt{18}}$

1.3 Exponents (indices)

Repeated multiplication of identical numbers can be written more efficiently by using exponential notation.

Exponential notation

If a is any real number ($a \in \mathbb{R}$) and n is a positive integer ($n \in \mathbb{Z}^+$), then

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$$

where n is the **exponent**, a is the **base** and a^n is called the ***n*th power** of a.
Note: n is also called the **power** or **index** (plural: indices).

Integer exponents

We now state seven laws of integer exponents (or indices) that you will have learned in a previous mathematics course. Familiarity with these rules is essential for work throughout this course.

Let a and b be real numbers ($a, b \in \mathbb{R}$) and let m and n be positive integers ($m, n \in \mathbb{Z}^+$). Assume that all denominators and bases are not equal to zero. All of the laws can be applied in either direction.

Table 1.5 Laws of exponents (indices) for integer exponents.

• **Hint:** It is important to recognise the difference between exponential expressions such as $(-3)^2$ and -3^2. In the expression $(-3)^2$, the parentheses make it clear that -3 is the base being raised to the power of 2. However, in -3^2 the negative sign is not considered to be a part of the base with the expression being the same as $-(3)^2$ so that 3 is the base being raised to the power of 2. Hence, $(-3)^2 = 9$ and $-3^2 = -9$.

	Property	Example	Description
1.	$b^m b^n = b^{m+n}$	$x^2x^5 = x^7$	multiplying like bases
2.	$\frac{b^m}{b^n} = b^{m-n}$	$\frac{2w^7}{3w^2} = \frac{2w^5}{3}$	dividing like bases
3.	$(b^m)^n = b^{mn}$	$(3^x)^2 = 3^{2x} = (3^2)^x = 9^x$	a power raised to a power
4.	$(ab)^n = a^n b^n$	$(4k)^3 = 4^3k^3 = 64k^3$	the power of a product
5.	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{y}{3}\right)^2 = \frac{y^2}{3^2} = \frac{y^2}{9}$	the power of a quotient
6.	$a^0 = 1$	$(t^2 + 5)^0 = 1$	definition of a zero exponent
7.	$a^{-n} = \frac{1}{a^n}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$	definition of a negative exponent

The last two laws of exponents listed above – the definition of a zero exponent and the definition of a negative exponent – are often assumed without proper explanation. The definition of a^n as repeated multiplication, i.e. n factors of a, is easily understood when n is a positive integer. So how do we formulate appropriate definitions for a^n when n is negative or zero? These definitions will have to be compatible with the laws for positive integer exponents. If the law stating $b^m b^n = b^{m+n}$ is to hold for a zero exponent, then $b^n b^0 = b^{n+0} = b^n$. Since the number 1 is the identity element for multiplication (multiplicative identity property) then $b^n \cdot 1 = b^n$. Therefore, we must define b^0 as the number 1. If the law $b^m b^n = b^{m+n}$ is to also hold for negative integer exponents, then $b^n b^{-n} = b^{n-n} = b^0 = 1$. Since the product of b^n and b^{-n} is 1, they must be reciprocals (multiplicative inverse property). Therefore, we must define b^{-n} as $\frac{1}{b^n}$.

Rational exponents

We know that $4^3 = 4 \times 4 \times 4$ and $4^0 = 1$ and $4^{-2} = \frac{1}{4^2} = \frac{1}{4 \times 4}$, but what meaning are we to give to $4^{\frac{1}{2}}$? In order to carry out algebraic operations with expressions having exponents that are rational numbers, it will be very helpful if they follow the laws established for integer exponents. From the law $b^m b^n = b^{m+n}$, it must follow that $4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\frac{1}{2}+\frac{1}{2}} = 4^1$. Likewise, from the law $(b^m)^n = b^{mn}$, it follows that $(4^{\frac{1}{2}})^2 = 4^{\frac{1}{2}\cdot 2} = 4^1$. Therefore, we need to define $4^{\frac{1}{2}}$ as the square root of 4 or, more precisely, as the principal (positive) square root of 4, that is, $\sqrt{4}$. We are now ready to use radicals to define a rational exponent of the form $\frac{1}{n}$, where n is a positive integer. If the rule $(b^m)^n = b^{mn}$ is to apply when $m = \frac{1}{n}$, it must follow that $(b^{\frac{1}{n}})^n = b^{\frac{n}{n}} = b^1$. This means that the nth power of $b^{\frac{1}{n}}$ is b and, from the discussion of nth roots in Section 1.2, we define $b^{\frac{1}{n}}$ as the principal nth root of b.

Definition of $b^{\frac{1}{n}}$

If $n \in \mathbb{Z}^+$, then $b^{\frac{1}{n}}$ is the principal nth root of b. Using a radical, this means

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

This definition allows us to evaluate exponential expressions such as the following:

$$36^{\frac{1}{2}} = \sqrt{36} = 6; (-27)^{\frac{1}{3}} = \sqrt[3]{-27} = -3; \left(\frac{1}{81}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{1}{81}} = \frac{1}{3}$$

Now we can apply the definition of $b^{\frac{1}{n}}$ and the rule $(b^m)^n = b^{mn}$ to develop a rule for expressions with exponents not just of the form $\frac{1}{n}$ but of the more general form $\frac{m}{n}$.

$$b^{\frac{m}{n}} = b^{m\frac{1}{n}} = (b^m)^{\frac{1}{n}} = \sqrt[n]{b^m}; \text{ or, equivalently, } b^{\frac{m}{n}} = b^{\frac{1}{n}m} = (b^{\frac{1}{n}})^m = (\sqrt[n]{b})^m$$

This will allow us to evaluate exponential expressions such as $9^{\frac{3}{2}}$, $(-8)^{\frac{5}{3}}$ and $64^{\frac{5}{6}}$.

Definition of rational exponents

If m and n are positive integers with no common factors, then

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} \text{ or } (\sqrt[n]{b})^m$$

If n is an even number, we must have $b \geqslant 0$.

The numerator of a rational exponent indicates the power to which the base of the exponential expression is raised, and the denominator indicates the root to be taken. With this definition for rational exponents, we can conclude that the laws of exponents, stated for integer exponents in Section 1.3, also hold true for rational exponents.

Example 4

Evaluate and/or simplify each of the following exponential expressions.

a) $(2xy^2)^3$ b) $2(xy^2)^3$ c) $(-2)^{-3}$

d) $(a-2)^0$ e) $(3^3)^{\frac{1}{2}} \cdot 9^{\frac{3}{4}}$ f) $\dfrac{a^{-2}b^4}{a^{-5}b^5}$

g) $(-32)^{-\frac{4}{5}}$ h) $8^{\frac{2}{3}}$ i) $\left(\frac{1}{2}x^2y\right)^3(x^3y^{-2})^{-1}$

j) $\dfrac{\sqrt{a+b}}{a+b}$ k) $\dfrac{(x+y)^2}{(x+y)^{-2}}$

Solution

a) $(2xy^2)^3 = 2^3x^3(y^2)^3 = 8x^3y^6$

b) $2(xy^2)^3 = 2x^3(y^2)^3 = 2x^3y^6$

c) $(-2)^{-3} = \dfrac{1}{(-2)^3} = -\dfrac{1}{8}$

d) $(a-2)^0 = 1$

e) $(3^3)^{\frac{1}{2}} \cdot 9^{\frac{3}{4}} = 3^{\frac{3}{2}}(3^2)^{\frac{3}{4}} = 3^{\frac{3}{2}} \cdot 3^{\frac{3}{2}} = 3^{\frac{6}{2}} = 3^3 = 27$

f) $\dfrac{a^{-2}b^4}{a^{-5}b^5} = \dfrac{a^{-2-(-5)}}{b^{5-4}} = \dfrac{a^3}{b}$

g) $(-32)^{-\frac{4}{5}} = [-2^5]^{-\frac{4}{5}} = (-2)^{-4} = \dfrac{1}{(-2)^4} = \dfrac{1}{16}$

h) $8^{\frac{2}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$ or $8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = (2)^2 = 4$ or $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$

i) $\left(\frac{1}{2}x^2y\right)^3(x^3y^{-2})^{-1} = \left(\dfrac{x^6y^3}{8}\right)(x^{-3}y^2) = \dfrac{x^{6-3}y^{3+2}}{8} = \dfrac{x^3y^5}{8}$

j) $\dfrac{\sqrt{a+b}}{a+b} = \dfrac{(a+b)^{\frac{1}{2}}}{(a+b)^1} = \dfrac{1}{(a+b)^{1-\frac{1}{2}}} = \dfrac{1}{(a+b)^{\frac{1}{2}}} = \dfrac{1}{\sqrt{a+b}}$

Note: Avoid an error here. $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$. Also, $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ and $\sqrt{a^2+b^2} \neq a+b$.

k) $\dfrac{(x+y)^2}{(x+y)^{-2}} = (x+y)^{2-(-2)} = (x+y)^4$

Note: Avoid an error here. $(x+y)^n \neq x^n + y^n$.

Although $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$, expanding is not generally 'simplifying'.

Exercise 1.3

In questions 1–6, simplify (without your GDC) each expression to a single integer.

1 $16^{\frac{1}{4}}$ **2** $9^{\frac{3}{2}}$ **3** $64^{\frac{2}{3}}$

4 $8^{\frac{4}{3}}$ **5** $32^{\frac{3}{5}}$ **6** $(\sqrt{2})^6$

In questions 7–9, simplify each expression (without your GDC) to a quotient of two integers.

7 $\left(\frac{8}{27}\right)^{\frac{2}{3}}$ **8** $\left(\frac{9}{16}\right)^{\frac{1}{2}}$ **9** $\left(\frac{25}{4}\right)^{\frac{3}{2}}$

In questions 10–13, evaluate (without your GDC) each expression.

10 $(-3)^{-2}$ **11** $(13)^0$ **12** $\dfrac{4 \cdot 3^{-2}}{2^{-2} \cdot 3^{-1}}$ **13** $\left(-\frac{3}{4}\right)^{-3}$

In questions 14–28, simplify each exponential expression (leave only positive exponents).

14 $3(-ab^2)^2$ **15** $3(-ab^2)^3$ **16** $(-3ab^2)^2$

17 $5x^3y^{-2} \cdot 2x^2y^5$ **18** $\dfrac{32w^2}{24w^3}$ **19** $\dfrac{6m^3n^{-2}}{8m^{-3}n^2}$

20 $\left(\frac{1}{2}m^2n^{-2}\right)^3$ **21** $3^{2m} \cdot 3^n$ **22** $\dfrac{x^{-1}y^5}{xy^3}$

23 $\dfrac{4a^3b^5}{(2a^2b)^4}$ **24** $\dfrac{(\sqrt[3]{x})(\sqrt[3]{x^4})}{\sqrt[3]{x^2}}$ **25** $\dfrac{12(a+b)^3}{9(a+b)}$

26 $\dfrac{(x+4y)^{\frac{1}{2}}}{2(x+4y)^{-1}}$ **27** $\dfrac{p^2+q^2}{\sqrt{p^2+q^2}}$ **28** $4^{3n} \cdot 2^{2m}$

1.4 Scientific notation (standard form)

Exponents provide an efficient way of writing and calculating with very large or very small numbers. The need for this is especially great in science. For example, a light year (the distance that light travels in one year) is 9 460 730 472 581 kilometres, and the mass of a single water molecule is 0.000 000 000 000 000 000 000 0056 grams. It is far more convenient and useful to write such numbers in **scientific notation** (also called **standard form**).

Definition of scientific notation

A positive number N is written in scientific notation if it is expressed in the form:

$$N = a \times 10^k \text{, where } 1 \leqslant a < 10 \text{ and } k \text{ is an integer}$$

In scientific notation, a light year is about 9.46×10^{12} kilometres. This expression is determined by observing that when a number is multiplied by 10^k and k is **positive**, the decimal point will move k places to the **right**. Therefore, $9.46 \times 10^{12} = 9\underbrace{460\,000\,000\,000}_{12 \text{ decimal places}}$. Knowing that when a number is multiplied by 10^k and k is **negative** the decimal point will move k places to the **left** helps us to express the mass of a water molecule as 5.6×10^{-24} grams. This expression is equivalent to $0.\underbrace{000\,000\,000\,000\,000\,000\,000\,0056}_{24 \text{ decimal places}}$.

Scientific notation is also a very convenient way of indicating the number of **significant figures** (digits) to which a number has been approximated. A light year expressed to an accuracy of 13 significant figures is 9 460 730 472 581 kilometres. However, many calculations will not require such a high degree of accuracy. For a certain calculation it may be more appropriate to have a light year approximated to 4 significant figures, which could be written as 9 461 000 000 000 kilometres, or more efficiently and clearly in scientific notation as 9.461×10^{12} kilometres.

Not only is scientific notation conveniently compact, it also allows a quick comparison of the magnitude of two numbers without the need to count zeros. Moreover, it enables us to use the laws of exponents to simplify otherwise unwieldy calculations.

Example 5

Use scientific notation to calculate each of the following.

a) $64\,000 \times 2\,500\,000\,000$

b) $\dfrac{0.000\,000\,78}{0.000\,000\,0012}$

c) $\sqrt[3]{27\,000\,000\,000}$

Solution

a) $64\,000 \times 2\,5000\,000\,000 = (6.4 \times 10^4)(2.5 \times 10^9)$

$= 6.4 \times 2.5 \times 10^4 \times 10^9$

$= 16 \times 10^{4+9}$

$= 1.6 \times 10^1 \times 10^{13} = 1.6 \times 10^{14}$

b) $\dfrac{0.000\,000\,78}{0.000\,000\,0012} = \dfrac{7.8 \times 10^{-7}}{1.2 \times 10^{-9}} = \dfrac{7.8}{1.2} \times \dfrac{10^{-7}}{10^{-9}} = 6.5 \times 10^{-7-(-9)}$

$= 6.5 \times 10^2$ or 650

c) $\sqrt[3]{27\,000\,000\,000} = (2.7 \times 10^{10})^{\frac{1}{3}} = (27 \times 10^9)^{\frac{1}{3}} = (27)^{\frac{1}{3}}(10^9)^{\frac{1}{3}}$

$= 3 \times 10^3$ or 3000

Your GDC will automatically express numbers in scientific notation when a large or small number exceeds its display range. For example, if you use your GDC to compute 2 raised to the 64th power, the display (depending on the GDC model) will show the approximation

`1.844674407E19` or `1.844674407 19`

The final digits indicate the power of 10, and we interpret the result as $1.844\,674\,408 \times 10^{19}$. ($2^{64}$ is exactly 18 446 744 073 709 551 616.)

Exercise 1.4

In questions 1–8, write each number in scientific notation, rounding to 3 significant figures.

1 253.8 **2** 0.007 81 **3** 7 405 239

4 0.000 001 0448 **5** 4.9812 **6** 0.001 991

7 Land area of Earth: 148 940 000 square kilometres

8 Relative density of hydrogen: 0.000 0899 grams per cm^3

In questions 9–12, write each number in ordinary decimal notation.

9 2.7×10^{-3} **10** 5×10^{7} **11** 9.035×10^{-8} **12** 4.18×10^{12}

In questions 13–16, use scientific notation and the laws of exponents to perform the indicated operations. Give the result in scientific notation rounded to 2 significant figures.

13 $(2.5 \times 10^{-3})(10 \times 10^{5})$ **14** $\dfrac{3.2 \times 10^{6}}{1.6 \times 10^{2}}$

15 $\dfrac{(1 \times 10^{-3})(3.28 \times 10^{6})}{4 \times 10^{7}}$ **16** $(2 \times 10^{3})^{4}(3.5 \times 10^{5})$

1.5 Algebraic expressions

Examples of algebraic expressions are:

$$5a^3b^2 \qquad 2x^2 + 7x - 8 \qquad \frac{y^3 - 1}{y + 1} \qquad \frac{(bx + c)^3}{2 - \sqrt{a}}$$

Algebraic expressions are formed by combining variables and constants using addition, subtraction, multiplication, division, exponents and radicals.

Polynomials

An algebraic expression that has only non-negative powers of one or more variable and contains no variable in a denominator is called a **polynomial.**

Definition of a polynomial in the variable *x*
Given $a_0, a_1, a_2, \ldots, a_n \in \mathbb{R}$ $a_n \neq 0$ and $n \in \mathbb{Z}^+$, a **polynomial in *x*** is a sum of distinct **terms** in the form

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

where $a_1, a_2, \ldots, a_n$ are the **coefficients**, a_0 is the **constant term** and n (the highest exponent) is the **degree** of the polynomial.

• **Hint:** Polynomials with one, two and three terms are called **monomials**, **binomials** and **trinomials**, respectively. A polynomial of degree one is called linear; degree two is called quadratic; degree three is cubic; and degree four is quartic. Quadratic equations and functions are covered in Chapter 2.

Polynomials are added or subtracted using the properties of real numbers that were discussed in Section 1.1. We do this by combining like terms – terms containing the same variable(s) raised to the same power(s) – and applying the distributive property.

For example,

$2x^2y + 6x^2 - 7x^2y = 2x^2y - 7x^2y + 6x^2$ rearranging terms so the like terms are together

$= (2 - 7)x^2y + 6x^2$ applying distributive property: $ab + ac = (b + c)a$

$= -5x^2y + 6x^2$ no like terms remain, so polynomial is simplified

Expanding and factorizing polynomials

We apply the distributive property in the other direction, i.e. $a(b + c) = ab + ac$, in order to multiply polynomials. For example,

$(2x - 3)(x + 5) = 2x(x + 5) - 3(x + 5)$

$= 2x^2 + 10x - 3x - 15$ collecting like terms $10x$ and $-3x$

$= 2x^2 + 7x - 15$ terms written in descending order of the exponents

The process of multiplying polynomials is often referred to as **expanding**. Especially in the case of a polynomial being raised to a power, the number of terms in the resulting polynomial, after applying the distributive property and combining like terms, has increased (expanded) compared to the original number of terms. For example,

$(x + 3)^2 = (x + 3)(x + 3)$ squaring a first degree (linear) binomial

$= x(x + 3) + 3(x + 3)$

$= x^2 + 3x + 3x + 9$

$= x^2 + 6x + 9$ the result is a second degree (quadratic) trinomial

and,

$(x + 1)^3 = (x + 1)(x + 1)(x + 1)$ cubing a first degree binomial

$= (x + 1)(x^2 + x + x + 1)$

$= x(x^2 + 2x + 1) + 1(x^2 + 2x + 1)$

$= x^3 + 2x^2 + x + x^2 + 2x + 1$

$= x^3 + 3x^2 + 3x + 1$ the result is a third degree (cubic) polynomial with four terms

A pair of binomials of the form $a + b$ and $a - b$ are called **conjugates**. In most instances, the product of two binomials produces a trinomial. However, the product of a pair of conjugates produces a binomial such that both terms are squares and the second term is negative – referred to as a **difference of two squares**. For example,

$(x + 5)(x - 5) = x(x - 5) + 5(x - 5)$ multiplying two conjugates

$= x^2 - 5x + 5x - 25$

$= x^2 - 25$ $x^2 - 25$ is a difference of two squares

The inverse (or undoing) of multiplication (expansion) is factorization. If it is helpful for us to rewrite a polynomial as a product, then we need to factorize it – i.e. apply the distributive property in the *reverse* direction. The previous four examples can be used to illustrate equivalent pairs of factorized and expanded polynomials.

Factorized		**Expanded**
$(2x-3)(x+5)$	$=$	$2x^2+7x-15$
$(x+3)^2$	$=$	x^2+6x+9
$(x+1)^3$	$=$	x^3+3x^2+3x+1
$(x+5)(x-5)$	$=$	x^2-25

Certain polynomial expansions (products) and factorizations occur so frequently you should be able to quickly recognize and apply them. Here is a list of some of the more common ones. You can verify these identities by performing the multiplication.

Common polynomial expansion and factorization patterns

Expanding →

$$(x+a)(x+b) = x^2+(a+b)x+ab$$
$$(ax+b)(cx+d) = acx^2+(ad+bc)x+bd$$
$$(a+b)(a-b) = a^2-b^2$$
$$(a+b)^2 = a^2+2ab+b^2$$
$$(a-b)^2 = a^2-2ab+b^2$$
$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$$
$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$$

← Factorizing

These identities are useful patterns into which we can substitute any number or algebraic expression for *a*, *b* or *x*. This allows us to efficiently find products and powers of polynomials and also to factorize many polynomials.

Example 6

Find each product.

a) $(x+2)(x-7)$ b) $(3x-4)(4x+1)$ c) $(6x+y)(6x-y)$

d) $(4h-5)^2$ e) $(x^2+2)^3$ f) $(3+2\sqrt{5})(3-2\sqrt{5})$

Solution

• **Hint:** You should be able to perform the middle step 'mentally' without writing it.

a) This product fits the pattern $(x+a)(x+b)=x^2+(a+b)x+ab$.

$$(x+2)(x-7)=x^2+(2-7)x+(2)(-7)=x^2-5x-14$$

b) This product fits the pattern $(ax+b)(cx+d)=acx^2+(ad+bc)x+bd$.

$$(3x-4)(4x+1)=12x^2+(3-16)x-4=12x^2-13x-4$$

c) This fits the pattern $(a+b)(a-b) = a^2 - b^2$ where the result is a difference of two squares.

$$(5x^3 + 3y)(5x^3 - 3y) = (5x^3)^2 - (3y)^2 = 25x^6 - 9y^2$$

d) This fits the pattern $(a-b)^2 = a^2 - 2ab + b^2$.

$$(4h-5)^2 = (4h)^2 - 2(4h)(5) + (5)^2 = 16h^2 - 40h + 25$$

e) This fits the pattern $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

$$(x^2+2)^3 = (x^2)^3 + 3(x^2)^2(2) + 3(x^2)(2)^2 + (2)^3 = x^6 + 6x^4 + 12x^2 + 8$$

f) The pair of expressions being multiplied do not have a variable but they are conjugates, so they fit the pattern $(a+b)(a-b) = a^2 - b^2$.

$$(3+2\sqrt{5})(3-2\sqrt{5}) = (3)^2 - (2\sqrt{5})^2 = 9 - (4 \cdot 5) = 9 - 20 = -11$$

Note: The result of multiplying two **irrational** conjugates is a single **rational** number. We will make use of this result to simplify certain fractions.

Example 7

Completely factorize the following expressions.

a) $2x^2 - 14x + 24$

b) $2x^2 + x - 15$

c) $4x^6 - 9$

d) $3y^3 + 24y^2 + 48y$

e) $(x+3)^2 - y^2$

f) $5x^3y + 20xy^3$

Solution

a) $2x^2 - 14x + 24$

$= 2(x^2 - 7x + 12)$ — factor out the greatest common factor

$= 2[x^2 + (-3-4)x + (-3)(-4)]$ — fits the pattern $(x+a)(x+b) = x^2 + (a+b)x + ab$

$= 2(x-3)(x-4)$ — 'trial and error' to find $-3 - 4 = -7$ and $(-3)(-4) = 12$

b) The terms have no common factor and the leading coefficient is not equal to one. This factorization requires a logical 'trial and error' approach. There are eight possible factorizations.

$(2x-1)(x+15)$ $(2x-3)(x+5)$ $(2x-5)(x+3)$ $(2x-15)(x+1)$

$(2x+1)(x-15)$ $(2x+3)(x-5)$ $(2x+5)(x-3)$ $(2x+15)(x-1)$

Testing the middle term in each, you find that the correct factorization is $2x^2 + x - 15 = (2x-5)(x+3)$.

c) This binomial can be written as the difference of two squares, $4x^6 - 9 = (2x^3)^2 - (3)^2$, fitting the pattern $a^2 - b^2 = (a+b)(a-b)$. Therefore, $4x^6 - 9 = (2x^3+3)(2x^3-3)$.

d) $3y^3 + 24y^2 + 48y = 3y(y^2 + 8y + 16)$ factor out the greatest common factor

$= 3y(y^2 + 2 \cdot 4y + 4^2)$ fits the pattern $a^2 + 2ab + b^2 = (a + b)^2$

$= 3y(y + 4)^2$

e) Fits the difference of two squares pattern: $a^2 - b^2 = (a + b)(a - b)$ with $a = x + 3$ and $b = y$.

Therefore, $(x + 3)^2 - y^2 = [(x + 3) + y][(x + 3) - y]$

$= (x + y + 3)(x - y + 3)$

(f) $5x^3y + 20xy^3 = 5xy(x^2 + 4y^2)$: although both of the terms x^2 and $4y^2$ are perfect squares, the expression $x^2 + 4y^2$ is not a *difference* of squares and, hence, it cannot be factorized. The sum of two squares, $a^2 + b^2$, cannot be factorized.

Guidelines for factoring polynomials

1. Factor out the greatest common factor, if one exists.
2. Determine if the polynomial, or any factors, fit any of the special polynomial patterns – and factor accordingly.
3. Any quadratic trinomial of the form $ax^2 + bx + c$ will require a logical trial and error approach, if it factorizes.

Most polynomials cannot be factored into a product of polynomials with integer coefficients. In fact, factoring is often difficult, even when possible, for polynomials with degree 3 or higher. Nevertheless, factorizing is a powerful algebraic technique that can be applied in many situations.

Algebraic fractions

An **algebraic fraction** (or rational expression) is a quotient of two algebraic expressions or two polynomials. Given a certain algebraic fraction, we must assume that the variable can only have values such that the denominator is not zero. For example, for the algebraic fraction $\frac{x+3}{x^2-4}$, x cannot be 2 or -2. Most of the algebraic fractions that we will encounter will have numerators and denominators that are polynomials.

Simplifying algebraic fractions

When trying to simplify algebraic fractions, we need to completely factor the numerator and denominator and cancel any common factors.

Example 8

Simplify each algebraic fraction.

a) $\frac{2a^2 - 2ab}{6ab - 6b^2}$ b) $\frac{1 - x^2}{x^2 + x - 2}$ c) $\frac{(x + h)^2 - x^2}{h}$

Solution

a) $\dfrac{2a^2-2ab}{6ab-6b^2}=\dfrac{2a(a-b)}{6b(a-b)}=\dfrac{{}^{1}2a}{{}_{3}6b}=\dfrac{a}{3b}$

b) $\dfrac{1-x^2}{x^2+x-2}=\dfrac{(1-x)(1+x)}{(x-1)(x+2)}=\dfrac{-(-1+x)(1+x)}{(x-1)(x+2)}=\dfrac{-(x-1)(x+1)}{(x-1)(x+2)}$

$=-\dfrac{x+1}{x+2}$ or $\dfrac{-x-1}{x+2}$

c) $\dfrac{(x+h)^2-x^2}{h}=\dfrac{x^2+2hx+h^2-x^2}{h}=\dfrac{2hx+h^2}{h}=\dfrac{h(2x+h)}{h}=2x+h$

Adding and subtracting algebraic fractions

Before any fractions – numerical or algebraic – can be added or subtracted they must be expressed with the same denominator, preferably the least common denominator. Then the numerators can be added or subtracted according to the rule: $\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}$.

Example 9

Perform the indicated operation and simplify.

a) $x-\frac{1}{x}$ b) $\dfrac{2}{a+b}+\dfrac{3}{a-b}$ c) $\dfrac{2}{x+2}-\dfrac{x-4}{2x^2+x-6}$

Solution

a) $x-\frac{1}{x}=\frac{x}{1}-\frac{1}{x}=\frac{x^2}{x}-\frac{1}{x}=\frac{x^2-1}{x}$ or $\dfrac{(x+1)(x-1)}{x}$

b) $\dfrac{2}{a+b}+\dfrac{3}{a-b}=\dfrac{2}{a+b}\cdot\dfrac{a-b}{a-b}+\dfrac{3}{a-b}\cdot\dfrac{a+b}{a+b}=\dfrac{2(a-b)+3(a+b)}{(a+b)(a-b)}$

$=\dfrac{2a-2b+3a+3b}{a^2-b^2}=\dfrac{5a+b}{a^2-b^2}$

c) $\dfrac{2}{x+2}-\dfrac{x-4}{2x^2+x-6}=\dfrac{2}{x+2}-\dfrac{x-4}{(2x-3)(x+2)}$

$=\dfrac{2}{x+2}\cdot\dfrac{2x-3}{2x-3}-\dfrac{x-4}{(2x-3)(x+2)}$

$=\dfrac{2(2x-3)-(x-4)}{(2x-3)(x+2)}$

$=\dfrac{4x-6-x+4}{(2x-3)(x+2)}$

$=\dfrac{3x-2}{(2x-3)(x+2)}$ or $\dfrac{3x-2}{2x^2+x-6}$

• **Hint:** Although it is true that $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$, be careful to avoid an error here: $\frac{a}{b+c}\neq\frac{a}{b}+\frac{a}{c}$. Also, be sure to only cancel common *factors* between numerator and denominator. It is true that $\frac{ac}{bc}=\frac{a}{b}$ (with the common factor of c cancelling) because $\frac{ac}{bc}=\frac{a}{b}\cdot\frac{c}{c}=\frac{a}{b}\cdot1=\frac{a}{b}$; but, in general, it is not true that $\frac{a+c}{b+c}=\frac{a}{b}$. c is not a common factor of the numerator and denominator.

Simplifying a compound fraction

Fractional expressions with fractions in the numerator or denominator, or both, are usually referred to as compound fractions. A compound fraction is best simplified by first simplifying both its numerator and denominator into single fractions, and then multiplying the numerator and denominator by the reciprocal of the denominator, i.e. $\dfrac{\frac{a}{b}}{\frac{c}{d}}=\dfrac{\frac{a}{b}\cdot\frac{d}{c}}{\frac{c}{d}\cdot\frac{d}{c}}=\dfrac{\frac{ad}{bc}}{1}=\dfrac{ad}{bc}$; thereby expressing the compound fraction as a single fraction.

Example 10

Simplify each compound fraction.

a) $\dfrac{\frac{1}{x+h}-\frac{1}{x}}{h}$ b) $\dfrac{\frac{a}{b}+1}{1-\frac{a}{b}}$ c) $\dfrac{x(1-2x)^{-\frac{3}{2}}+(1-2x)^{-\frac{1}{2}}}{1-x}$

Solution

a) $$\frac{\frac{1}{x+h}-\frac{1}{x}}{h}=\frac{\frac{x}{x(x+h)}-\frac{x+h}{x(x+h)}}{\frac{h}{1}}=\frac{\frac{x-(x+h)}{x(x+h)}}{\frac{h}{1}}=\frac{x-x-h}{x(x+h)}\cdot\frac{1}{h}$$
$$=\frac{-\cancel{h}}{x(x+h)}\cdot\frac{1}{\cancel{h}}=-\frac{1}{x(x+h)}$$

b) $$\frac{\frac{a}{b}+1}{1-\frac{a}{b}}=\frac{\frac{a}{b}+\frac{b}{b}}{\frac{b}{b}-\frac{a}{b}}=\frac{\frac{a+b}{b}}{\frac{b-a}{b}}=\frac{a+b}{\cancel{b}}\cdot\frac{\cancel{b}}{b-a}=\frac{a+b}{b-a}$$

- **Hint:** Factor out the power of $1-2x$ with the *smallest* exponent.

c) $$\frac{x(1-2x)^{-\frac{3}{2}}+(1-2x)^{-\frac{1}{2}}}{1-x}=\frac{(1-2x)^{-\frac{3}{2}}[x+(1-2x)^{1}]}{1-x}$$
$$=\frac{(1-2x)^{-\frac{3}{2}}[x+1-2x]}{1-x}$$
$$=\frac{(1-2x)^{-\frac{3}{2}}\cancel{(1-x)}}{\cancel{1-x}}$$
$$=\frac{1}{(1-2x)^{\frac{3}{2}}}$$

With rules for rational exponents and radicals we can do the following, but it's not any *simpler*...

$$\frac{1}{(1-2x)^{\frac{3}{2}}}=\frac{1}{\sqrt{(3x-2)^{3}}}=\frac{1}{\sqrt{(3x-2)^{2}}\sqrt{3x-2}}=\frac{1}{|3x-2|\sqrt{3x-2}}$$

Rationalizing the denominator

Recall Example 3 from Section 1.2 where we rationalized the denominator of the numerical fractions $\frac{2}{\sqrt{3}}$ and $\frac{\sqrt{7}}{4\sqrt{10}}$. Also recall from earlier in this section that expressions of the form $a+b$ and $a-b$ are called conjugates and their product is a^2-b^2 (difference of two squares). If a fraction has an irrational denominator of the form $a+b\sqrt{c}$, we can change it to a rational expression ('rationalize') by multiplying numerator and denominator by its conjugate $a-b\sqrt{c}$, given that $(a+b\sqrt{c})(a-b\sqrt{c})=a^2-(b\sqrt{c})^2=a^2-b^2c$.

Example 11

Rationalize the denominator of each fractional expression.

a) $\dfrac{2}{1+\sqrt{5}}$ b) $\dfrac{1}{\sqrt{x}+1}$

Solution

a) $$\frac{2}{1+\sqrt{5}}=\frac{2}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}=\frac{2(1-\sqrt{5})}{1-(\sqrt{5})^2}=\frac{2(1-\sqrt{5})}{1-5}=\frac{\cancel{2}(1-\sqrt{5})}{-_{2}\cancel{4}}$$
$$=\frac{-(1-\sqrt{5})}{2}=\frac{-1+\sqrt{5}}{2}$$

b) $$\frac{1}{\sqrt{x}+1}=\frac{1}{\sqrt{x}+1}\cdot\frac{\sqrt{x}-1}{\sqrt{x}-1}=\frac{\sqrt{x}-1}{(\sqrt{x})^2-1^2}=\frac{\sqrt{x}-1}{x-1}$$

Exercise 1.5

In questions 1–12, expand and simplify.

1 $(n+4)(n-5)$

2 $(2y-3)(5y+3)$

3 $(x+7)(x-7)$

4 $(5m+2)^2$

5 $(x-1)^3$

6 $(1+\sqrt{a})(1-\sqrt{a})$

7 $(a+b)(a-b+1)$

8 $[(2x+3)+y][(2x+3)-y]$

9 $(a+b)^3$

10 $(ax+b)^2$

11 $(1+\sqrt{5})(1-\sqrt{5})$

12 $(2x-1)(2x^2-3x+5)$

In questions 13–30, completely factorize the expression.

13 $12x^2-48$

14 x^3-6x^2

15 x^2+x-12

16 $7-6m-m^2$

17 $x^2-10x+16$

18 y^2+7y+6

19 $3n^2-21n+30$

20 $2x^3+20x^2+18x$

21 a^2-16

22 $3y^2-14y-5$

23 $25n^4-4$

24 $ax^2+6ax+9a$

25 $2n(m+1)^2-(m+1)^2$

26 x^4-1

27 $9-(y-3)^2$

28 $4y^4-10y^3-96y^2$

29 $4x^2-20x+25$

30 $(2x+3)^{-2}+2x(2x+3)^{-3}$

In questions 31–36, simplify the algebraic fraction.

31 $\dfrac{x+4}{x^2+5x+4}$

32 $\dfrac{3n-3}{6n^2-6n}$

33 $\dfrac{a^2-b^2}{5a-5b}$

34 $\dfrac{x^2+4x+4}{x+2}$

35 $\dfrac{2a-5}{5-2a}$

36 $\dfrac{(2x+h)^2-4x^2}{h}$

In questions 37–46, perform the indicated operation and simplify.

37 $\dfrac{x}{5}-\dfrac{x-1}{3}$

38 $\dfrac{1}{a}-\dfrac{1}{b}$

39 $\dfrac{2}{2x-1}-4$

40 $\dfrac{x}{x+3}+\dfrac{1}{x}$

41 $\dfrac{1}{x+y}+\dfrac{1}{x-y}$

42 $\dfrac{3}{x-2}+\dfrac{5}{2-x}$

43 $\dfrac{2x-6}{x}\cdot\dfrac{3x}{x-3}$

44 $\dfrac{3}{y+2}+\dfrac{5}{y^2-3y-10}$

45 $\dfrac{a+b}{b}\cdot\dfrac{1}{a^2-b^2}$

46 $\dfrac{3x^2-3}{6x}\cdot\dfrac{5x^2}{1-x}$

In questions 47–50, rationalize the denominator of each fractional expression.

47 $\dfrac{1}{3-\sqrt{2}}$

48 $\dfrac{5}{2+\sqrt{3}}$

49 $\dfrac{2\sqrt{2}+\sqrt{3}}{2\sqrt{2}-\sqrt{3}}$

50 $\dfrac{1}{\sqrt{5}+7}$

1.6 Equations and formulae

Equations, identities and formulae

We will encounter a wide variety of equations in this course. Essentially an equation is a statement equating two algebraic expressions that may be true or false depending upon what value(s) is/are substituted for the variable(s). The value(s) of the variable(s) that make the equation true are called the **solutions** or **roots** of the equation. All of the solutions to an equation comprise the **solution set** of the equation. An equation that is true for all possible values of the variable is called an **identity**. All of the common polynomial expansion and factorization patterns shown in Section 1.5 are identities. For example, $(a + b)^2 = a^2 + 2ab + b^2$ is true for all values of a and b. The following are also examples of identities.

$$3(x - 5) = 2(x + 3) + x - 21 \qquad (x + y)^2 - 2xy = x^2 + y^2$$

Many equations are often referred to as a **formula** (plural: formulae) and typically contain more than one variable and, often, other symbols that represent specific constants or **parameters** (constants that may change in value but do not alter the properties of the expression). Formulae with which you are familiar include:
$A = \pi r^2$, $d = rt$, $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ and $V = \frac{4}{3}\pi r^3$

Whereas most equations that we will encounter will have numerical solutions, we can solve a formula for a certain variable in terms of other variables – sometimes referred to as changing the subject of a formula.

One of the most famous equations in the history of mathematics, $x^n + y^n = z^n$, is associated with Pierre Fermat (1601–1665), a French lawyer and amateur mathematician. Writing in the margin of a French translation of *Arithmetica*, Fermat conjectured that the equation $x^n + y^n = z^n$ $(x, y, z, n \in \mathbb{Z})$ has no non-zero solutions for the variables x, y and z when the parameter n is greater than two. When $n = 2$, the equation is equivalent to Pythagoras' theorem for which there are an infinite number of integer solutions – Pythagorean triples, such as $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$, and their multiples. Fermat claimed to have a proof for his conjecture but that he could not fit it in the margin. All the other margin conjectures in Fermat's copy of *Arithmetica* were proven by the start of the 19th century, but this one remained unproven for over 350 years, until the English mathematician Andrew Wiles proved it in 1994.

Example 12

Solve for the indicated variable in each formula.

a) $a^2 + b^2 = c^2$ solve for b

b) $T = 2\pi\sqrt{\frac{l}{g}}$ solve for l

Solution

a) $a^2 + b^2 = c^2 \Rightarrow b^2 = c^2 - a^2 \Rightarrow b = \pm\sqrt{c^2 - a^2}$
If b is a length then $b = \sqrt{c^2 - a^2}$.

b) $T = 2\pi\sqrt{\frac{l}{g}} \Rightarrow \sqrt{\frac{l}{g}} = \frac{T}{2\pi} \Rightarrow \frac{l}{g} = \frac{T^2}{4\pi^2} \Rightarrow l = \frac{T^2 g}{4\pi^2}$

The graph of an equation

Two important characteristics of any equation are the number of variables (unknowns) and the type of algebraic expressions it contains (e.g. polynomials, rational expressions, trigonometric, exponential, etc.). Nearly all of the equations in this course will have either one or two variables, and in this introductory chapter we will discuss only equations with algebraic expressions that are polynomials. Solutions for equations with a single variable will consist of individual numbers that can be *graphed* as points on a number line. The **graph** of an equation is a visual representation of the equation's solution set. For example, the solution set of the one-variable equation containing quadratic and linear polynomials $x^2 = 2x + 8$ is $x \in \{-2, 4\}$. The graph of this one-variable equation is depicted (Figure 1.6) on a one-dimensional coordinate system, i.e. the real number line.

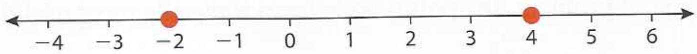


Figure 1.6 The solution set.

The solution set of a two-variable equation will be an **ordered pair** of numbers. An ordered pair corresponds to a location indicated by a point on a two-dimensional coordinate system, i.e. a **coordinate plane.** For example, the solution set of the two-variable **quadratic equation** $y = x^2$ will be an infinite set of ordered pairs (x, y) that satisfy the equation. (Quadratic equations will be covered in detail in Chapter 2.)

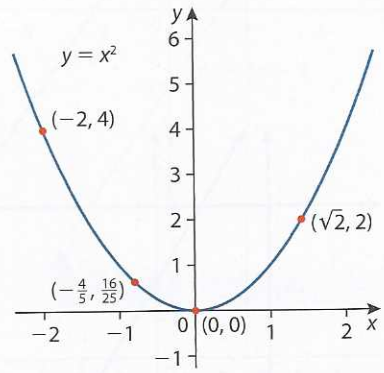


Figure 1.7 Four ordered pairs in the solution set of $y = x^2$ are graphed in red. The graph of all the ordered pairs in the solution set form a curve, as shown in blue.

Equations of lines

A one-variable **linear equation** in x can always be written in the form $ax + b = 0$, $a \neq 0$, and it will have exactly one solution, $x = -\frac{b}{a}$. An example of a two-variable **linear equation in x and y** is $x - 2y = 2$. The graph of this equation's solution set (an infinite set of ordered pairs) is a **line.** (See Figure 1.8.)

The **slope** m, or **gradient,** of a non-vertical line is defined by the formula $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{vertical change}}{\text{horizontal change}}$. Because division by zero is undefined, the slope of a vertical line is undefined. Using the two points $(1, -\frac{1}{2})$ and $(4, 1)$, we compute the slope of the line with equation $x - 2y = 2$ to be

$$m = \frac{1 - (-\frac{1}{2})}{4 - 1} = \frac{\frac{3}{2}}{\frac{3}{1}} = \frac{1}{2}.$$

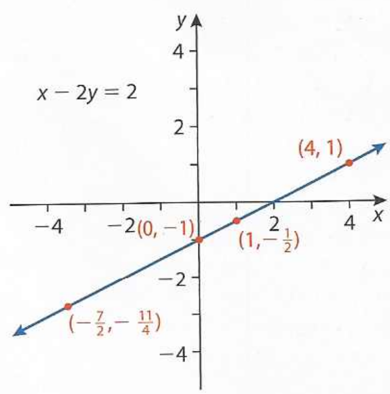


Figure 1.8 The graph of $x - 2y = 2$.

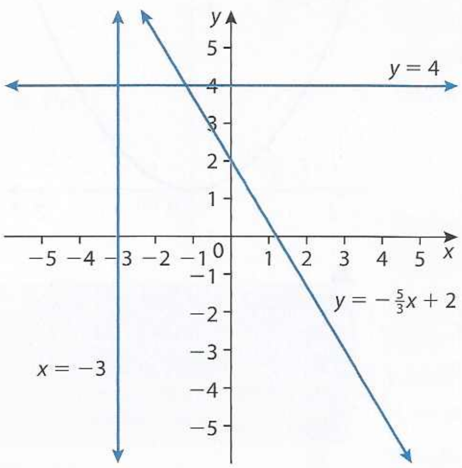
If we solve for y, we can rewrite the equation in the form $y = \frac{1}{2}x - 1$. Note that the coefficient of x is the slope of the line and the constant term is the y-coordinate of the point at which the line intersects the y-axis, i.e. the y-intercept. There are several forms in which to write linear equations whose graphs are lines.

Form	Equation	Characteristics
general form	$ax + by + c = 0$	every line has an equation in this form if both a and $b \neq 0$
slope-intercept form	$y = mx + c$	m is the slope; $(0, c)$ is the y-intercept
point-slope form	$y - y_1 = m(x - x_1)$	m is the slope; (x_1, y_1) is a known point on the line
horizontal line	$y = c$	slope is zero; $(0, c)$ is the y-intercept
vertical line	$x = c$	slope is undefined; unless line is y-axis, no y-intercept

Table 1.6 Forms for equations of lines.

Most problems involving equations and graphs fall into two categories: (1) given an equation, determine its graph; and (2) given a graph, or some information about it, find its equation. For lines, the first type of problem is often best solved by using the slope-intercept form. However, for the second type of problem, the point-slope form is usually most useful.

Example 13



Without using a GDC, sketch the line that is the graph of each of the following linear equations, written here in general form.

a) $5x + 3y - 6 = 0$
b) $y - 4 = 0$
c) $x + 3 = 0$

Solution

a) Solve for y to write the equation in slope-intercept form.
$5x + 3y - 6 = 0 \Rightarrow 3y = -5x + 6 \Rightarrow y = -\frac{5}{3}x + 2$. The line has a y-intercept of $(0, 2)$ and a slope of $-\frac{5}{3}$.

b) The equation $y - 4 = 0$ is equivalent to $y = 4$, whose graph is a horizontal line with a y-intercept of $(0, 4)$.

c) The equation $x + 3 = 0$ is equivalent to $x = -3$, whose graph is a vertical line with no y-intercept; but, it has an x-intercept of $(-3, 0)$.

Example 14

a) Find the equation of the line that passes through the point $(3, 31)$ and has a slope of 12. Write the equation in slope-intercept form.

b) Find the linear equation in C and F knowing that when $C = 10$ then $F = 50$, and when $C = 100$ then $F = 212$. Solve for F in terms of C.

Solution

a) Substitute into the point-slope form $y - y_1 = m(x - x_1)$; $x_1 = 3$, $y_1 = 31$ and $m = 12$

$$y - y_1 = m(x - x_1) \Rightarrow y - 31 = 12(x - 3) \Rightarrow y = 12x - 36 + 31 \Rightarrow y = 12x - 5$$

b) The two points, ordered pairs (C, F), that are known to be on the line are $(10, 50)$ and $(100, 212)$. The variable C corresponds to the variable x and F corresponds to y in the definitions and forms stated above. The slope of the line is $m = \frac{F_2 - F_1}{C_2 - C_1} = \frac{212 - 50}{100 - 10} = \frac{162}{90} = \frac{9}{5}$. Choose one of the points on the line, say $(10, 50)$, and substitute it and the slope into the point-slope form.

$$F - F_1 = m(C - C_1) \Rightarrow F - 50 = \frac{9}{5}(C - 10) \Rightarrow F = \frac{9}{5}C - 18 + 50 \Rightarrow F = \frac{9}{5}C + 32$$

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular.

The two lines graphed in Figure 1.9 suggest the following property: Two distinct non-vertical lines are **parallel** if, and only if, their slopes are equal, $m_1 = m_2$.

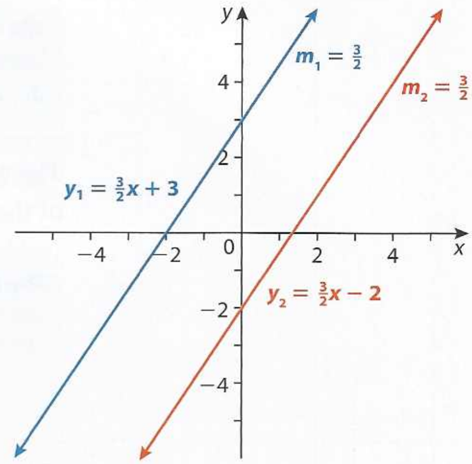


Figure 1.9

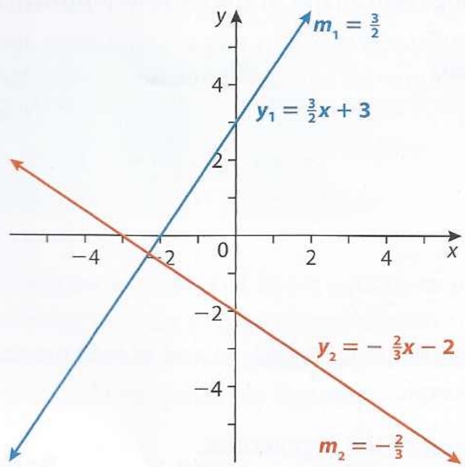


Figure 1.10

The two lines graphed in Figure 1.10 suggest another property: Two non-vertical lines are **perpendicular** if, and only if, their slopes are negative reciprocals – that is, $m_1 = -\frac{1}{m_2}$, which is equivalent to $m_1 \cdot m_2 = -1$.

Distances and midpoints

Recall from Section 1.1 that absolute value (modulus) is used to define the **distance** (always positive) between two points on the real number line.

The distance between the points A and B on the real number line is $|B - A|$, which is equivalent to $|A - B|$.

The points A and B are the endpoints of a line segment that is denoted with the notation $[AB]$ and the length of the line segment is denoted AB. In Figure 1.11, the distance between A and B is $AB = |4 - (-2)| = |-2 - 4| = 6$.

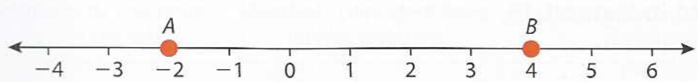


Figure 1.11

The distance between two general points (x_1, y_1) and (x_2, y_2) on a coordinate plane can be found using the definition for distance on a number line and Pythagoras' theorem. For the points (x_1, y_1) and (x_2, y_2), the horizontal distance between them is $|x_1 - x_2|$ and the vertical distance is $|y_1 - y_2|$. As illustrated in Figure 1.12, these distances are the lengths of two legs of a right-angled triangle whose hypotenuse is the distance between the points. If d represents the distance between (x_1, y_1) and (x_2, y_2), then by Pythagoras' theorem $d^2 = |x_1 - x_2|^2 + |y_1 - y_2|^2$. Because the square of any number is positive, the absolute value is not necessary, giving us the distance formula for two-dimensional coordinates.

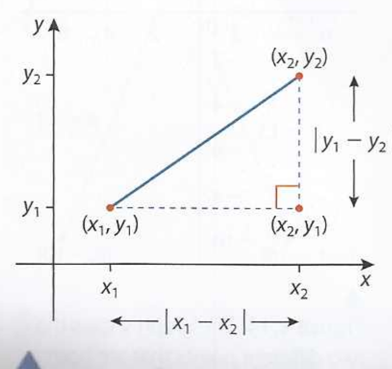


Figure 1.12

The distance formula

The distance d between the two points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

The coordinates of the **midpoint** of a line segment are the average values of the corresponding coordinates of the two endpoints.

The midpoint formula

The midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) in the coordinate plane is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Example 15

a) Show that the points $P(1, 2)$, $Q(3, 1)$ and $R(4, 8)$ are the vertices of a right-angled triangle.

b) Find the midpoint of the hypotenuse.

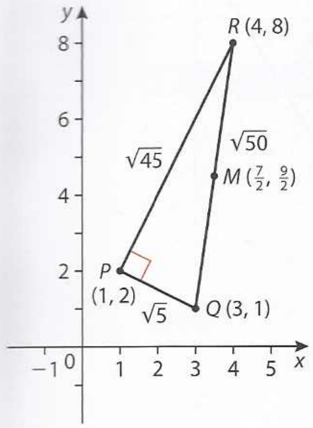


Figure 1.13

Solution

a) The three points are plotted and the line segments joining them are drawn in Figure 1.13. Applying the distance formula, we can find the exact lengths of the three sides of the triangle.

$$PQ = \sqrt{(1-3)^2 + (2-1)^2} = \sqrt{4+1} = \sqrt{5}$$

$$QR = \sqrt{(3-4)^2 + (1-8)^2} = \sqrt{1+49} = \sqrt{50}$$

$$PR = \sqrt{(1-4)^2 + (2-8)^2} = \sqrt{9+36} = \sqrt{45}$$

$PQ^2 + PR^2 = QR^2$ because $(\sqrt{5})^2 + (\sqrt{45})^2 = 5 + 45 = 50 = (\sqrt{50})^2$. The lengths of the three sides of the triangle satisfy Pythagoras' theorem, confirming that the triangle is a right-angled triangle.

b) QR is the hypotenuse. Let the midpoint of QR be point M. Using the midpoint formula, $M = \left(\frac{3+4}{2}\right), \left(\frac{1+8}{2}\right) = \left(\frac{7}{2}, \frac{9}{2}\right)$. This point is plotted in Figure 1.13.

Example 16

Find x so that the distance between the points $(1, 2)$ and $(x, -10)$ is 13.

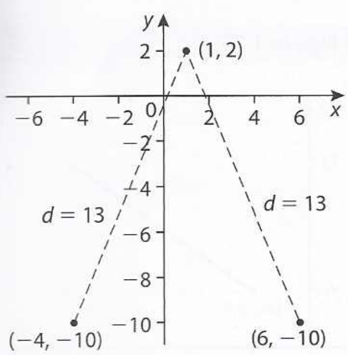


Figure 1.14 The graph shows the two different points that are both a distance of 13 from (1, 2).

Solution

$d = 13 = \sqrt{(x-1)^2 + (-10-2)^2} \Rightarrow 13^2 = (x-1)^2 + (-12)^2$

$\Rightarrow 169 = x^2 - 2x + 1 + 144 \Rightarrow x^2 - 2x - 24 = 0$

$\Rightarrow (x-6)(x+4) = 0 \Rightarrow x - 6 = 0$ or $x + 4 = 0$

$\Rightarrow x = 6$ or $x = -4$

Simultaneous equations

Many problems that we solve with algebraic techniques involve sets of equations with several variables, rather than just a single equation with one or two variables. Such a set of equations is called a set of **simultaneous equations** because we find the values for the variables that solve all of the equations simultaneously. In this section, we consider only the simplest set of simultaneous equations – a pair of linear equations in two variables. We will take a brief look at three methods for solving simultaneous linear equations. They are:

1. Graphical method
2. Elimination method
3. Substitution method

Although we will only look at pairs of linear equations in this section, it is worthwhile mentioning that the graphical and substitution methods are effective for solving sets of equations where not all of the equations are linear, e.g. one linear and one quadratic equation.

Graphical method

The graph of each equation in a system of two linear equations in two unknowns is a line. The graphical interpretation of the solution of a pair of simultaneous linear equations corresponds to determining what point, or points, lies on both lines. Two lines in a coordinate plane can only relate to one another in one of three ways: (1) intersect at exactly one point, (2) intersect at all points on each line (i.e. the lines are identical), or (3) the two lines do not intersect (i.e. the lines are parallel). These three possibilities are illustrated in Figure 1.15.

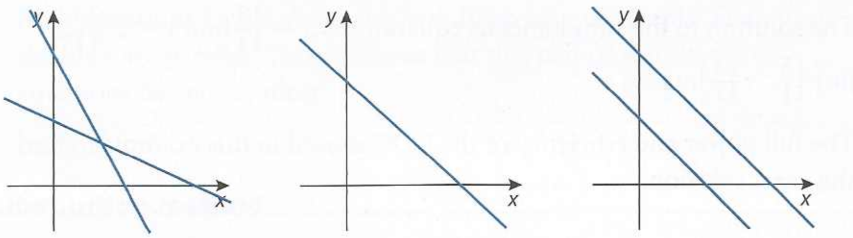


Figure 1.15

Intersect at exactly one point; exactly one solution — Identical – coincident lines; infinite solutions — Never intersect – parallel lines; no solution

Although a graphical approach to solving simultaneous linear equations provides a helpful visual picture of the number and location of solutions, it can be tedious and inaccurate if done by hand. The graphical method is far more efficient and accurate when performed on a graphical display calculator (GDC).

Example 17

Use the graphical features of a GDC to solve each pair of simultaneous equations.

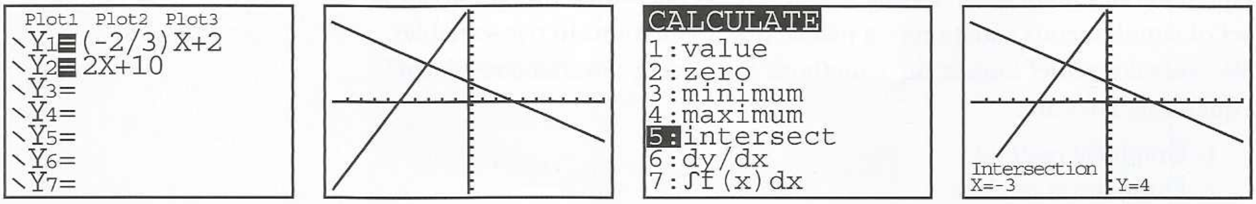
a) $2x + 3y = 6$
   $2x - y = -10$

b) $7x - 5y = 20$
   $3x + y = 2$

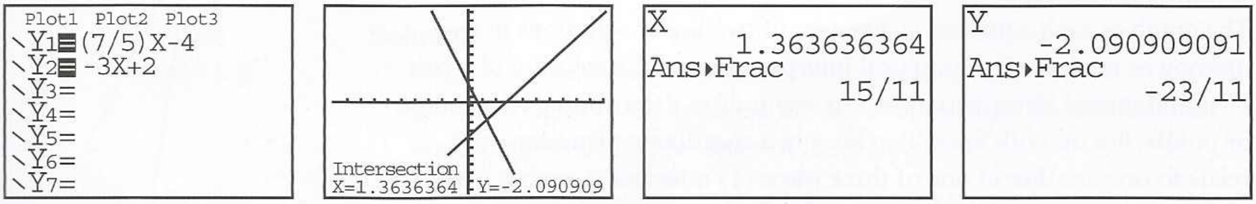
Solution

a) First, we will rewrite each equation in slope-intercept form, i.e. $y = mx + c$. This is a necessity if we use our GDC, and is also very useful for graphing by hand (manual).

$$2x + 3y = 6 \Rightarrow 3y = -2x + 6 \Rightarrow y = -\frac{2}{3}x + 2 \text{ and } 2x - y = -10 \Rightarrow y = 2x + 10$$



The intersection point and solution to the simultaneous equations is $x = -3$ and $y = 4$, or $(-3, 4)$. If we manually graphed the two linear equations in a) very carefully using graph paper, we may have been able to determine the exact coordinates of the intersection point. However, using a graphical method without a GDC to solve the simultaneous equations in b) would only allow us to crudely approximate the solution.



b) $7x - 5y = 20 \Rightarrow 5y = 7x - 20 \Rightarrow y = \frac{7}{5}x - 4$ and
$3x + y = 2 \Rightarrow y = -3x + 2$

The solution to the simultaneous equations is $x = \frac{15}{11}$ and $y = -\frac{23}{11}$, or $\left(\frac{15}{11}, -\frac{23}{11}\right)$.

The full power and efficiency of the GDC is used in this example to find the exact solution.

Elimination method

To solve a system using the **elimination method**, we try to combine the two linear equations using sums or differences in order to eliminate one of the variables. Before combining the equations, we need to multiply one or both of the equations by a suitable constant to produce coefficients for one of the variables that are equal (then subtract the equations), or that differ only in sign (then add the equations).

Example 18

Use the elimination method to solve each pair of simultaneous equations.

a) $5x + 3y = 9$
$2x - 4y = 14$

b) $x - 2y = 3$
$2x - 4y = 5$

Solution

a) We can obtain coefficients for y that differ only in sign by multiplying the first equation by 4 and the second equation by 3. Then we add the equations to eliminate the variable y.

$$\begin{array}{rcl} 5x + 3y = 9 & \rightarrow & 20x + 12y = 36 \\ \underline{2x - 4y = 14} & \rightarrow & \underline{6x - 12y = 42} \\ & & 26x = 78 \\ & & x = \frac{78}{26} \\ & & x = 3 \end{array}$$

By substituting the value of 3 for x in either of the original equations we can solve for y.

$$5x + 3y = 9 \Rightarrow 5(3) + 3y = 9 \Rightarrow 3y = -6 \Rightarrow y = -2$$

The solution is $(3, -2)$.

b) To obtain coefficients for x that are equal, we multiply the first equation by 2 and then subtract the equations to eliminate the variable x.

$$\begin{array}{rcl} x - 2y = 7 & \rightarrow & 2x - 4y = 14 \\ \underline{2x - 4y = 5} & \rightarrow & \underline{2x - 4y = 5} \\ & & 0 = 9 \end{array}$$

Because it is not possible for 0 to equal 9, there is no solution. The lines that are the graphs of the two equations are parallel. To confirm this we can rewrite each of the equations in the form $y = mx + c$.

$$x - 2y = 7 \Rightarrow 2y = x - 7 \Rightarrow y = \tfrac{1}{2}x - \tfrac{7}{2} \text{ and}$$
$$2x - 4y = 5 \Rightarrow 4y = 2x - 5 \Rightarrow y = \tfrac{1}{2}x - \tfrac{5}{2}$$

Both equations have a slope of $\frac{1}{2}$, but different y-intercepts. Therefore, the lines are parallel. This confirms that this pair of simultaneous equations has no solution.

Substitution method

The algebraic method that can be applied effectively to the widest variety of simultaneous equations, including non-linear equations, is the **substitution method**. Using this method, we choose one of the equations and solve for one of the variables in terms of the other variable. We then substitute this expression into the other equation to produce an equation with only one variable, which we can solve directly.

Example 19

Use the substitution method to solve each pair of simultaneous equations.

a) $3x - y = -9$
$6x + 2y = 2$

b) $-2x + 6y = 4$
$3x - 9y = -6$

Solution

a) Solve for y in the top equation, $3x - y = -9 \Rightarrow y = 3x + 9$, and substitute $3x + 9$ in for y in the bottom equation:
$6x + 2(3x + 9) = 2 \Rightarrow 6x + 6x + 18 = 2 \Rightarrow 12x = -16 \Rightarrow x = -\frac{16}{12} = -\frac{4}{3}$.
Now substitute $-\frac{4}{3}$ for x in either equation to solve for y.
$3\left(-\frac{4}{3}\right) - y = -9 \Rightarrow y = -4 + 9 \Rightarrow y = 5$.
The solution is $x = -\frac{4}{3}, y = 5$, or $\left(-\frac{4}{3}, 5\right)$.

b) Solve for x in the top equation,
$-2x + 6y = 4 \Rightarrow 2x = 6y - 4 \Rightarrow x = 3y - 2$, and substitute $3y - 2$ in for x in the bottom equation:
$3(3y - 2) - 9y = -6 \Rightarrow 9y - 6 - 9y = -6 \Rightarrow 0 = 0$.
The resulting equation $0 = 0$ is true for any values of x and y. The two equations are equivalent, and their graphs will produce identical lines – i.e. coincident lines. Therefore, the solution set consists of all points (x, y) lying on the line $-2x + 6y = 4$ $\left(\text{or } y = \frac{1}{3}x + \frac{2}{3}\right)$.

Exercise 1.6

In questions 1–8, solve for the indicated variable in each formula.

1 $m(h - x) = n$ solve for x

2 $v = \sqrt{ab - t}$ solve for a

3 $A = \frac{h}{2}(b_1 + b_2)$ solve for b_1

4 $A = \frac{1}{2}r^2\theta$ solve for r

5 $\frac{f}{g} = \frac{h}{k}$ solve for k

6 $at = x - bt$ solve for t

7 $V = \frac{1}{3}\pi r^3 h$ solve for r

8 $F = \frac{g}{m_1 k + m_2 k}$ solve for k

In questions 9–12, find the equation of the line that passes through the two given points. Write the line in slope-intercept form ($y = mx + c$), if possible.

9 $(-9, 1)$ and $(3, -7)$

10 $(3, -4)$ and $(10, -4)$

11 $(-12, -9)$ and $(4, 11)$

12 $\left(\frac{7}{3}, -\frac{1}{2}\right)$ and $\left(\frac{7}{3}, \frac{5}{2}\right)$

13 Find the equation of the line that passes through the point $(7, -17)$ and is parallel to the line with equation $4x + y - 3 = 0$. Write the line in slope-intercept form ($y = mx + c$).

14 Find the equation of the line that passes through the point $\left(-5, \frac{11}{2}\right)$ and is perpendicular to the line with equation $2x - 5y - 35 = 0$. Write the line in slope-intercept form ($y = mx + c$).

In questions 15–18, a) find the exact distance between the points, and b) find the midpoint of the line segment joining the two points.

15 $(-4, 10)$ and $(4, -5)$

16 $(-1, 2)$ and $(5, 4)$

17 $\left(\frac{1}{2}, 1\right)$ and $\left(-\frac{5}{2}, \frac{4}{3}\right)$

18 $(12, 2)$ and $(-10, 9)$

In questions 19 and 20, find the value(s) of k so that the distance between the points is 5.

19 $(5, -1)$ and $(k, 2)$

20 $(-2, -7)$ and $(1, k)$

In questions 21–23, show that the given points form the vertices of the indicated polygon.

21 Right-angled triangle: $(4, 0)$, $(2, 1)$ and $(-1, -5)$

22 Isosceles triangle: $(1, -3)$, $(3, 2)$ and $(-2, 4)$

23 Parallelogram: $(0, 1)$, $(3, 7)$, $(4, 4)$ and $(1, -2)$

In questions 24–29, use the elimination method to solve each pair of simultaneous equations.

24 $x + 3y = 8$
$x - 2y = 3$

25 $x - 6y = 1$
$3x + 2y = 13$

26 $6x + 3y = 6$
$5x + 4y = -1$

27 $x + 3y = -1$
$x - 2y = 7$

28 $8x - 12y = 4$
$-2x + 3y = 2$

29 $5x + 7y = 9$
$-11x - 5y = 1$

In questions 30–35, use the substitution method to solve each pair of simultaneous equations.

30 $2x + y = 1$
$3x + 2y = 3$

31 $3x - 2y = 7$
$5x - y = -7$

32 $2x + 8y = -6$
$-5x - 20y = 15$

33 $\frac{x}{5} + \frac{y}{2} = 8$
$x + y = 20$

34 $2x - y = -2$
$4x + y = 5$

35 $0.4x + 0.3y = 1$
$0.25x + 0.1y = -0.25$

In questions 36–38, solve the pair of simultaneous equations using any method – elimination, substitution or the graphical features of your GDC.

36 $3x + 2y = 9$
$7x + 11y = 2$

37 $3.62x - 5.88y = -10.11$
$0.08x - 0.02y = 0.92$

38 $2x - 3y = 4$
$5x + 2y = 1$