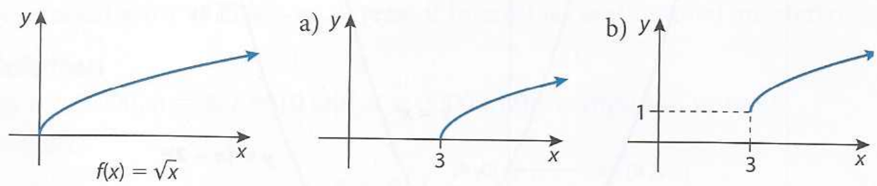
• **Hint:** A common error is caused by confusion about the direction of a horizontal translation since $f(x)$ is translated *left* if a *positive* number is added *inside* the argument of the function – e.g. $g(x) = (x + 3)^2$ is obtained by translating $f(x) = x$ three units *left*. You are in the habit of associating *positive* with movement to the *right* (as on the x-axis) instead of *left*. Whereas $f(x)$ is translated *up* if a *positive* number is added *outside* the function – e.g. $g(x) = x^2 + 3$ is obtained by translating $f(x) = x$ three units *up*. This agrees with the convention that a *positive* number is associated with an *upward* movement (as on the y-axis). An alternative (and more consistent) approach to vertical and horizontal translations is to think of what number is being added directly to the x- or y-coordinate. For example, the equation for the graph obtained by translating the graph of $y = x^2$ three units up is $y = x^2 + 3$, which can also be written as $y - 3 = x^2$. In this form, negative three is added to the y-coordinate (vertical coordinate), which causes a vertical translation in the *upward* (or positive) direction. Likewise, the equation for the graph obtained by translating the graph of $y = x^2$ two units to the right is $y = (x - 2)^2$. Negative two is added to the x-coordinate (horizontal coordinate), which causes a horizontal translation to the right (or positive direction). There is consistency between vertical and horizontal translations. Assuming that movement up or to the right is considered positive, and that movement down or to the left is negative, then the direction for either type of translation is opposite to the sign ($\pm$) of the number being added to the vertical (y) or horizontal (x) coordinate. In fact, what is actually being translated is the y-axis or the x-axis. For example, the graph of $y - 3 = x^2$ can also be obtained by not changing the graph of $y = x^2$ but instead translating the y-axis three units down – which creates exactly the same effect as translating the graph of $y = x^2$ three units up.

Example 18

The diagrams show how the graph of $y = \sqrt{x}$ is transformed to the graph of $y = f(x)$ in three steps. For each diagram, a) and b), give the equation of the curve.

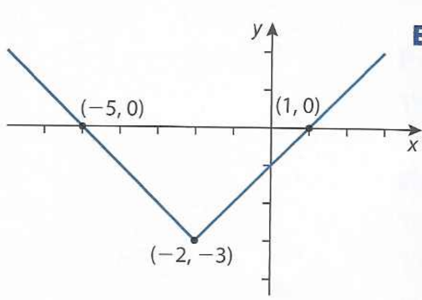


> Note that in Example 18, if the transformations had been performed in reverse order – that is, the vertical translation followed by the horizontal translation – it would produce the same final graph (in part b)) with the same equation. In other words, when applying both a vertical and horizontal translation on a function it does not make any difference which order they are applied (i.e. they are commutative). However, as we will see further on in the chapter, it *can* make a difference to how other sequences of transformations are applied. In general, transformations are *not* commutative.

Solution

To obtain graph a), the graph of $y = \sqrt{x}$ is translated three units to the right. To produce the equation of the translated graph, -3 is added *inside* the argument of the function $y = \sqrt{x}$. Therefore, the equation of the curve graphed in a) is $y = \sqrt{x - 3}$.

To obtain graph b), the graph of $y = \sqrt{x - 3}$ is translated up one unit. To produce the equation of the translated graph, $+1$ is added *outside* the function. Therefore, the equation of the curve graphed in b) is $y = \sqrt{x - 3} + 1$ (or $y = 1 + \sqrt{x - 3}$).



Example 19

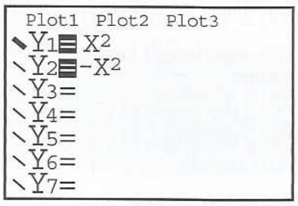
Write the equation of the absolute value function whose graph is shown on the left.

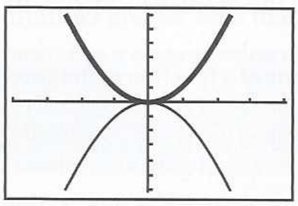
Solution

The graph shown is exactly the same shape as the graph of the equation $y = |x|$ but in a different position. Given that the vertex is $(-2, -3)$, it is clear that this graph can be obtained by translating $y = |x|$ two units left and then three units down. When we move $y = |x|$ two units left we get the graph of $y = |x + 2|$. Moving the graph of $y = |x + 2|$ three units down gives us the graph of $y = |x + 2| - 3$. Therefore, the equation of the graph shown is $y = |x + 2| - 3$. (Note: The two translations applied in reverse order produce the same result.)

Reflections

Use your GDC to graph the two functions $f(x) = x^2$ and $g(x) = -x^2$. The graph of $g(x) = -x^2$ is a reflection in the x-axis of $f(x) = x^2$. This certainly makes sense because g is formed by multiplying f by -1, causing the y-coordinate of each point on the graph of $y = -x^2$ to be the negative of the y-coordinate of the point on the graph of $y = x^2$ with the same x-coordinate.





Figures 2.20 and 2.21 illustrate that the graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis.

• **Hint:** The expression $-x^2$ is potentially ambiguous. It is accepted to be equivalent to $-(x)^2$. It is *not* equivalent to $(-x)^2$. For example, if you enter the expression -3^2 into your GDC, it gives a result of -9, *not* $+9$. In other words, the expression -3^2 is consistently interpreted as 3^2 being multiplied by -1. The same as $-x^2$ is interpreted as x^2 being multiplied by -1.

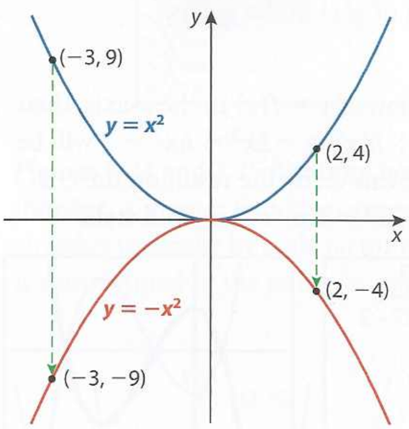


Figure 2.20

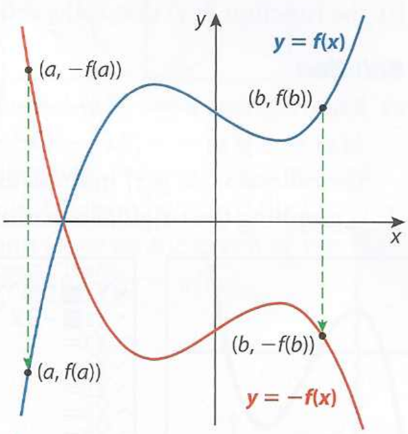


Figure 2.21

Graph the functions $f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt{-x - 2}$. Previously, with $f(x) = x^2$ and $g(x) = -x^2$, g was formed by multiplying the entire function f by -1. However, for $f(x) = \sqrt{x - 2}$ and $g(x) = \sqrt{-x - 2}$, g is formed by multiplying the variable x by -1. In this case, the graph of $g(x) = \sqrt{-x - 2}$ is a reflection in the y-axis of $f(x) = \sqrt{x - 2}$. This makes sense if you recognize that the y-coordinate on the graph of $y = \sqrt{-x}$ will be the same as the y-coordinate on the graph of $y = \sqrt{x}$, if the value substituted for x in $y = \sqrt{-x}$ is the opposite of the value of x in $y = \sqrt{x}$. For example, if $x = 9$ then $y = \sqrt{9} = 3$; and, if $x = -9$ then $y = \sqrt{-(-9)} = \sqrt{9} = 3$. Opposite values of x in the two functions produce the same y-coordinate for each.

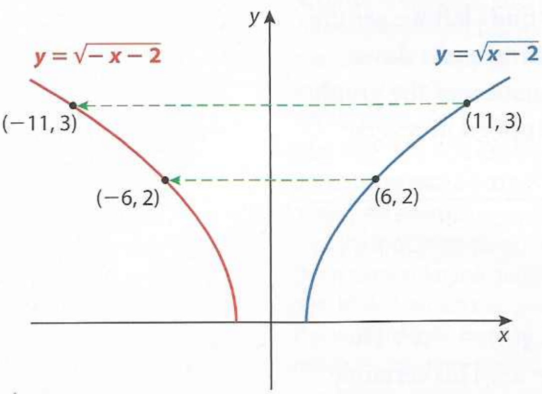


Figure 2.22

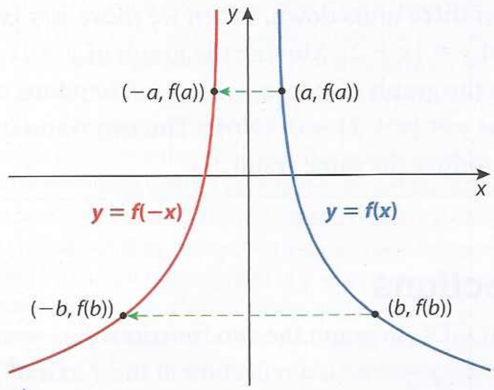


Figure 2.23

Figures 2.22 and 2.23 illustrate that the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

Reflections of a function in the coordinate axes

I. The graph of $y = -f(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis.
II. The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis.

Example 20

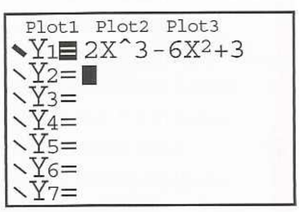
For $g(x) = 2x^3 - 6x^2 + 3$, find:

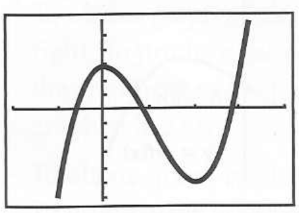
a) the function $h(x)$ that is the reflection of $g(x)$ in the x-axis

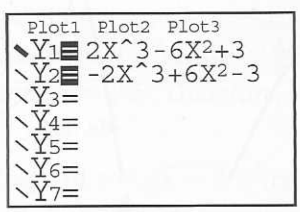
b) the function $p(x)$ that is the reflection of $g(x)$ in the y-axis.

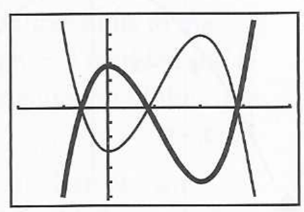
Solution

a) Knowing that $y = -f(x)$ is the reflection of $y = f(x)$ in the x-axis, then $h(x) = -g(x) = -(2x^3 - 6x^2 + 3) \Rightarrow h(x) = -2x^3 + 6x^2 - 3$ will be the reflection of $g(x)$ in the x-axis. We can verify the result on the GDC – graphing the original equation $y = 2x^3 - 6x^2 + 3$ in bold style.

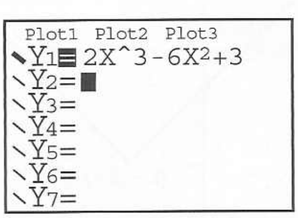


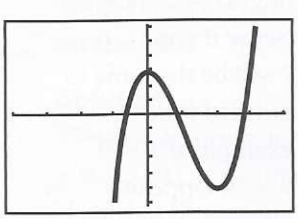


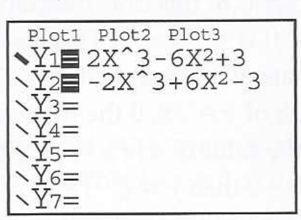


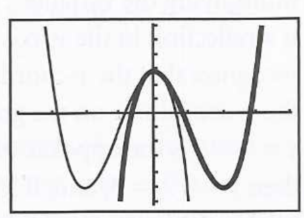


b) Knowing that $y = f(-x)$ is the reflection of $y = f(x)$ in the y-axis, we need to substitute $-x$ for x in $y = g(x)$. Thus, $p(x) = g(-x) = 2(-x)^3 - 6(-x)^2 + 3 \Rightarrow p(x) = -2x^3 - 6x + 3$ will be the reflection of $g(x)$ in the y-axis. Again, we can verify the result on the GDC – graphing the original equation $y = 2x^3 - 6x^2 + 3$ in bold style.







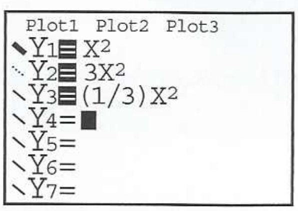


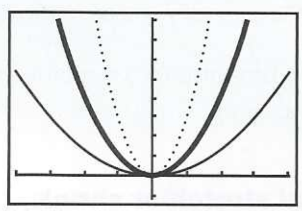
Non-rigid transformations: stretching and shrinking

Horizontal and vertical translations, and reflections in the x- and y-axes are called **rigid transformations** because the shape of the graph does not change – only its position is changed. **Non-rigid transformations** cause the shape of the original graph to change. The non-rigid transformations that we will study cause the shape of a graph to *stretch* or *shrink* in either the vertical or horizontal direction.

Vertical stretch or shrink

Graph the following three functions: $f(x) = x^2$, $g(x) = 3x^2$ and $h(x) = \frac{1}{3}x^2$. How do the graphs of g and h compare to the graph of f? Clearly, the shape of the graphs of g and h is not the same as the graph of f. Multiplying the function f by a positive number greater than one, or less than one, has distorted the shape of the graph. For a certain value of x, the y-coordinate of $y = 3x^2$ is three times the y-coordinate of $y = x^2$. Therefore, the graph of $y = 3x^2$ can be obtained by *vertically stretching* the graph of $y = x^2$ by a factor of 3 (**scale factor 3**). Likewise, the graph of $y = \frac{1}{3}x^2$ can be obtained by *vertically shrinking* the graph of $y = x^2$ by **scale factor** $\frac{1}{3}$.





Figures 2.24 and 2.25 illustrate how multiplying a function by a positive number, a, *greater than one* causes a transformation by which the function *stretches* vertically by scale factor a. A point (x, y) on the graph of $y = f(x)$ is transformed to the point (x, ay) on the graph of $y = af(x)$.

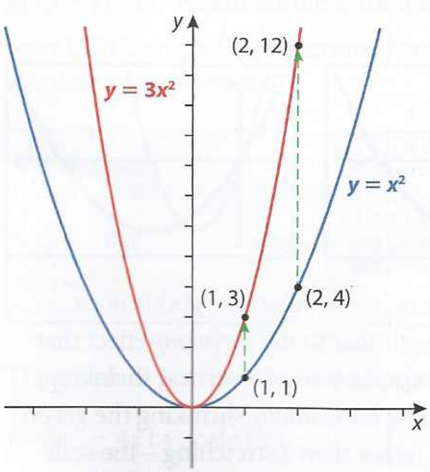


Figure 2.24

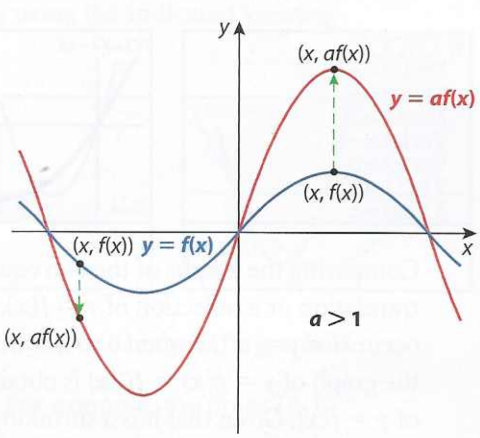


Figure 2.25

Figures 2.26 and 2.27 illustrate how multiplying a function by a positive number, a, *greater than zero and less than one* causes the function to *shrink* vertically by scale factor a. A point (x, y) on the graph of $y = f(x)$ is transformed to the point (x, ay) on the graph of $y = af(x)$.

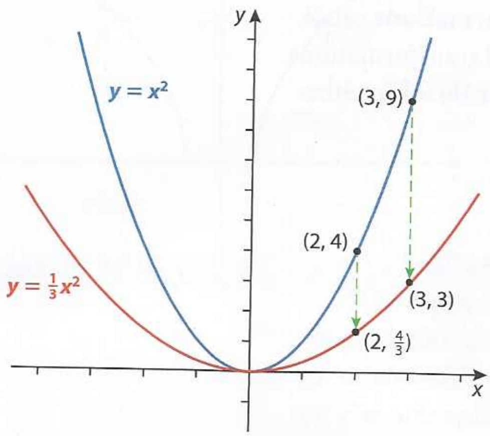


Figure 2.26

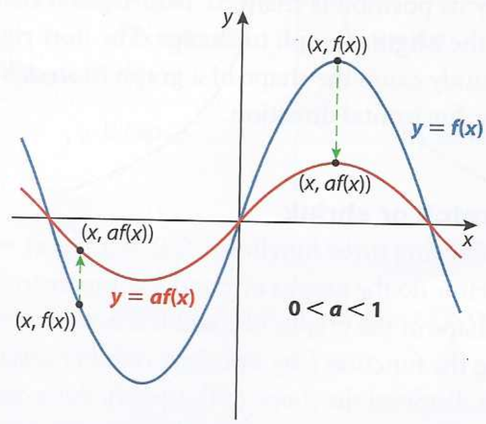


Figure 2.27

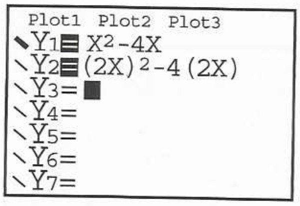
Vertical stretching and shrinking of functions

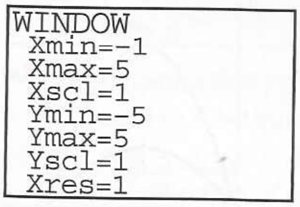
I. If $a > 1$, the graph of $y = af(x)$ is obtained by *vertically stretching* the graph of $y = f(x)$.

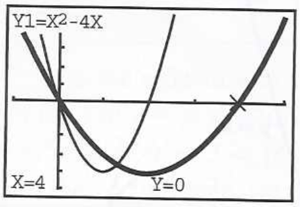
II. If $0 < a < 1$, the graph of $y = af(x)$ is obtained by *vertically shrinking* the graph of $y = f(x)$.

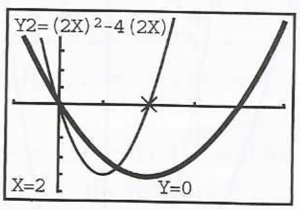
Horizontal stretch or shrink

Let's investigate how the graph of $y = f(ax)$ is obtained from the graph of $y = f(x)$. Given $f(x) = x^2 - 4x$, find another function, $g(x)$, such that $g(x) = f(2x)$. We substitute $2x$ for x in the function f, giving $g(x) = (2x)^2 - 4(2x)$. For the purposes of our investigation, let's leave $g(x)$ in this form. On your GDC, graph these two functions, $f(x) = x^2 - 4x$ and $g(x) = (2x)^2 - 4(2x)$, using the indicated viewing window and graphing f in bold style.









Comparing the graphs of the two equations, we see that $y = g(x)$ is *not* a translation or a reflection of $y = f(x)$. It is similar to the *shrinking* effect that occurs for $y = af(x)$ when $0 < a < 1$, except, instead of a vertical shrinking, the graph of $y = g(x) = f(2x)$ is obtained by *horizontally* shrinking the graph of $y = f(x)$. Given that it is a shrinking – rather than a stretching – the scale factor must be less than one. Consider the point $(4, 0)$ on the graph of $y = f(x)$. The point on the graph of $y = g(x) = f(2x)$ with the same y-coordinate and on the same side of the parabola is (2, 0). The x-coordinate of the point on $y = f(2x)$ is the x-coordinate of the point on $y = f(x)$ multiplied by $\frac{1}{2}$. Use your GDC to confirm this for other pairs of corresponding points on $y = x^2 - 4x$ and $y = (2x)^2 - 4(2x)$ that have the same y-coordinate. The graph of $y = f(2x)$ can be obtained by *horizontally shrinking* the graph of $y = f(x)$ by scale factor $\frac{1}{2}$. This makes sense because if $f(2x_2) = (2x_2)^2 - 4(2x_2)$ and $f(x_1) = x_1^2 - 4x_1$ are to produce the same y-value then $2x_2 = x_1$; and, thus, $x_2 = \frac{1}{2}x_1$. Figures 2.28 and 2.29 illustrate how multiplying the x-variable of a function by a positive number, a, *greater than one* causes the function to *shrink* horizontally by scale factor $\frac{1}{a}$. A point (x, y) on the graph of $y = f(x)$ is transformed to the point $\left(\frac{1}{a}x, y\right)$ on the graph of $y = f(ax)$.

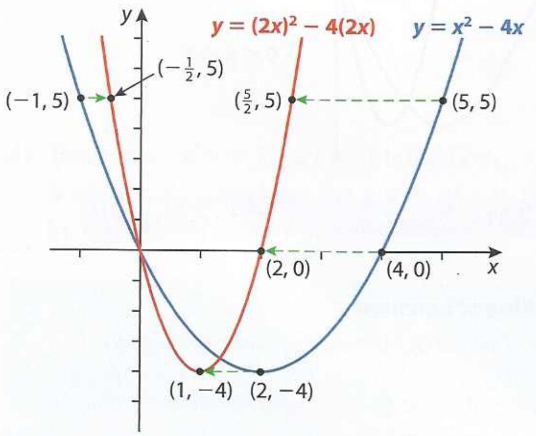


Figure 2.28

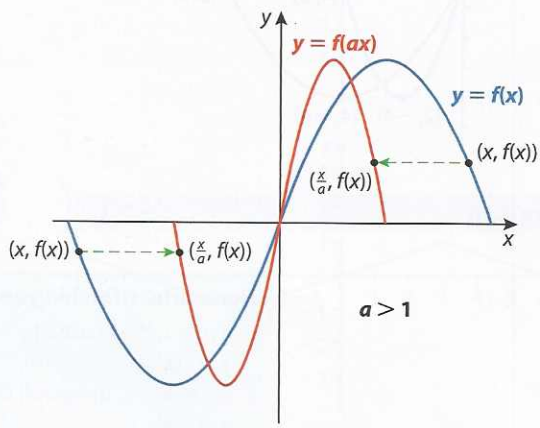
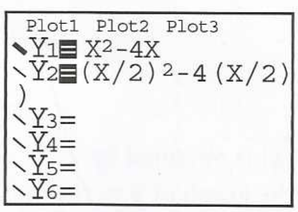
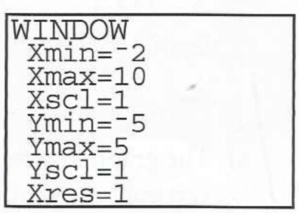
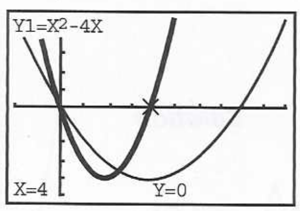


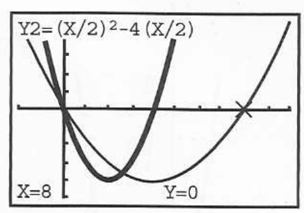
Figure 2.29

If $0 < a < 1$, the graph of the function $y = f(ax)$ is obtained by a *horizontal stretching* of the graph of $y = f(x)$ – rather than a shrinking – because the scale factor $\frac{1}{a}$ will be a value greater than 1 if $0 < a < 1$. Now, letting $a = \frac{1}{2}$ and, again using the function $f(x) = x^2 - 4x$, find $g(x)$, such that $g(x) = f\left(\frac{1}{2}x\right)$. We substitute $\frac{x}{2}$ for x in f, giving $g(x) = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$. On your GDC, graph the functions f and g using the indicated viewing window with f in bold.









The graph of $y = \left(\frac{x}{2}\right)^2 - 4\left(\frac{x}{2}\right)$ is a horizontal stretching of the graph of $y = x^2 - 4x$ by scale factor $\frac{1}{a} = \frac{1}{\frac{1}{2}} = 2$. For example, the point (4, 0) on $y = f(x)$ has been moved horizontally to the point (8, 0) on $y = g(x) = f\left(\frac{x}{2}\right)$.

Figures 2.30 and 2.31 illustrate how multiplying the x-variable of a function by a positive number, a, *greater than zero and less than one* causes the function to *stretch* horizontally by scale factor $\frac{1}{a}$. A point (x, y) on the graph of $y = f(x)$ is transformed to the point $\left(\frac{1}{a}x, y\right)$ on the graph of $y = f(ax)$.

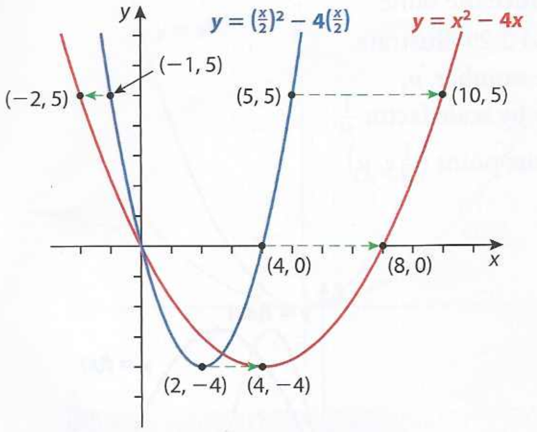


Figure 2.30

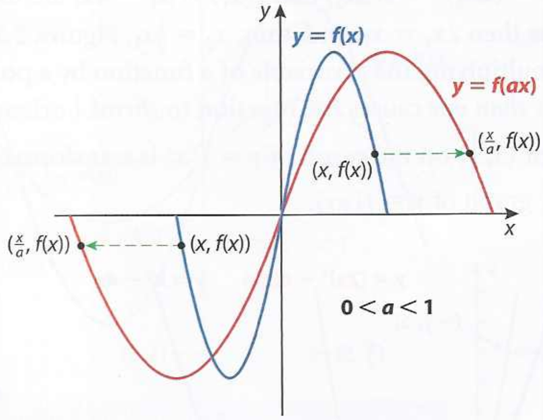


Figure 2.31

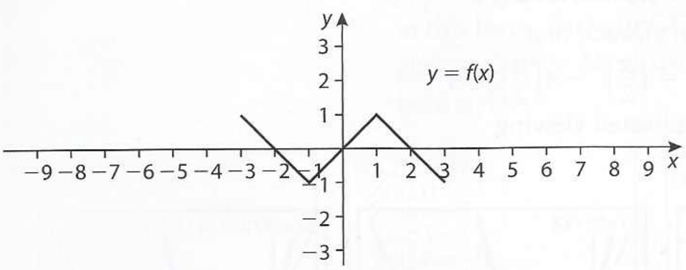
Horizontal stretching and shrinking of functions

I. If $a > 1$, the graph of $y = f(ax)$ is obtained by *horizontally shrinking* the graph of $y = f(x)$.

II. If $0 < a < 1$, the graph of $y = f(ax)$ is obtained by *horizontally stretching* the graph of $y = f(x)$.

Example 21

The graph of $y = f(x)$ is shown. Sketch the graph of each of the following two functions.



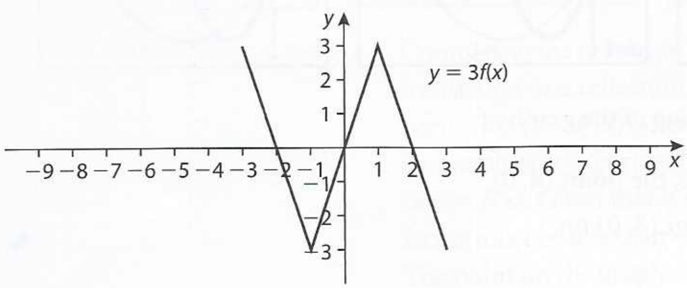
a) $y = 3f(x)$

b) $y = \frac{1}{3}f(x)$

c) $y = f(3x)$

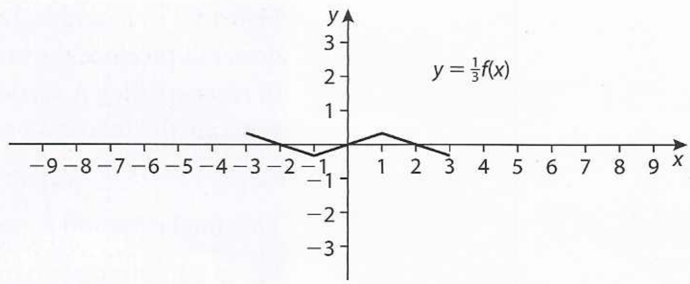
d) $y = f\left(\frac{1}{3}x\right)$

Solution

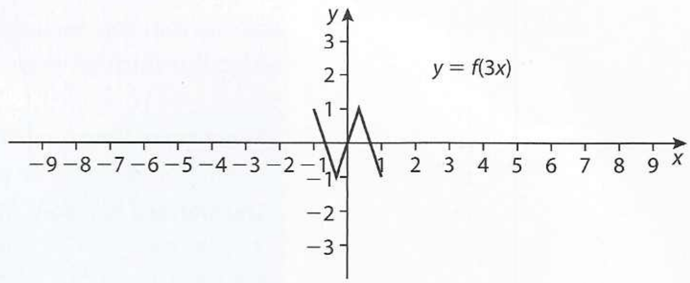


a) The graph of $y = 3f(x)$ is obtained by vertically stretching the graph of $y = f(x)$ by scale factor 3.

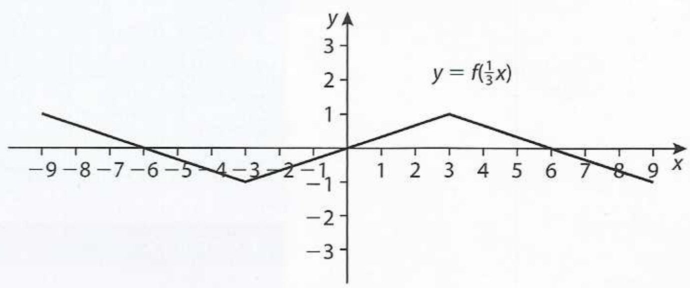
b) The graph of $y = \frac{1}{3}f(x)$ is obtained by vertically shrinking the graph of $y = f(x)$ by scale factor $\frac{1}{3}$.



c) The graph of $y = f(3x)$ is obtained by horizontally shrinking the graph of $y = f(x)$ by scale factor $\frac{1}{3}$.



d) The graph of $y = f\left(\frac{1}{3}x\right)$ is obtained by horizontally stretching the graph of $y = f(x)$ by scale factor 3.



Example 22

Describe the sequence of transformations performed on the graph of $y = x^2$ to obtain the graph of $y = 4x^2 - 3$.

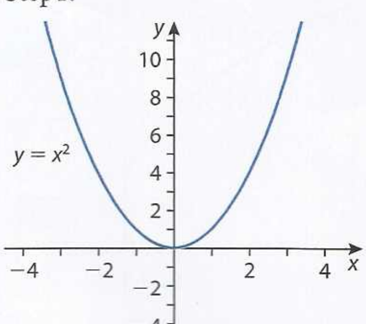
Solution

Step 1: Start with the graph of $y = x^2$.

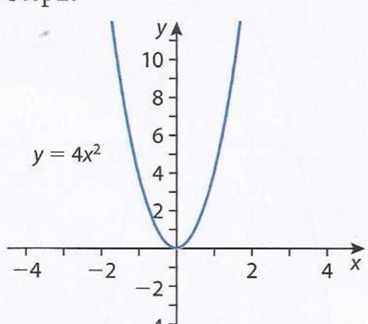
Step 2: Vertically stretch $y = x^2$ by scale factor 4.

Step 3: Vertically translate $y = 4x^2$ three units down.

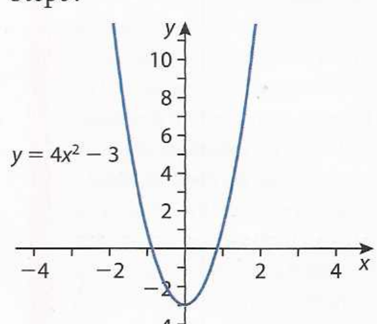
Step1:



Step2:



Step3:



Note that in Example 22, a vertical stretch followed by a vertical translation does not produce the same graph if the two transformations are performed in reverse order. A vertical translation followed by a vertical stretch would generate the following sequence of equations:

Step1: $y = x^2$ Step 2: $y = x^2 - 3$ Step 3: $y = 4(x^2 - 3) = 4x^2 - 12$

This final equation is not the same as $y = 4x^2 - 3$.

When combining two or more transformations, the order in which they are performed can make a difference. In general, when a sequence of transformations includes a vertical/horizontal stretch or shrink, or a reflection through the x-axis, the order may make a difference.

Summary of transformations on the graphs of functions

Assume that a, h and k are positive real numbers.

Transformed function	Transformation performed on $y = f(x)$
$y = f(x) + k$	vertical translation k units up
$y = f(x) - k$	vertical translation k units down
$y = f(x - h)$	horizontal translation h units right
$y = f(x + h)$	horizontal translation h units left
$y = -f(x)$	reflection in the x-axis
$y = f(-x)$	reflection in the y-axis
$y = af(x)$	vertical stretch ($a > 1$) or shrink ($0 < a < 1$)
$y = f(ax)$	horizontal stretch ($0 < a < 1$) or shrink ($a > 1$)

Exercise 2.4

In questions 1–14, sketch the graph of f, without a GDC or by plotting points, by using your knowledge of some of the basic functions shown in Figure 2.15.

1 $f: x \mapsto x^2 - 6$ **2** $f: x \mapsto (x - 6)^2$ **3** $f: x \mapsto |x| + 4$

4 $f: x \mapsto |x + 4|$ **5** $f: x \mapsto 5 + \sqrt{x - 2}$ **6** $f: x \mapsto \frac{1}{x - 3}$

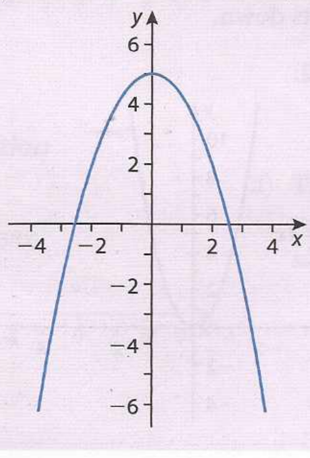
7 $f: x \mapsto \frac{1}{(x + 5)^2} + 2$ **8** $f: x \mapsto -x^3 - 4$ **9** $f: x \mapsto -|x - 1| + 6$

10 $f: x \mapsto \sqrt{-x + 3}$ **11** $f: x \mapsto 3\sqrt{x}$ **12** $f: x \mapsto \frac{1}{2}x^2$

13 $f: x \mapsto \left(\frac{1}{2}x\right)^2$ **14** $f: x \mapsto (-x)^3$

In questions 15–18, write the equation for the graph that is shown.

15



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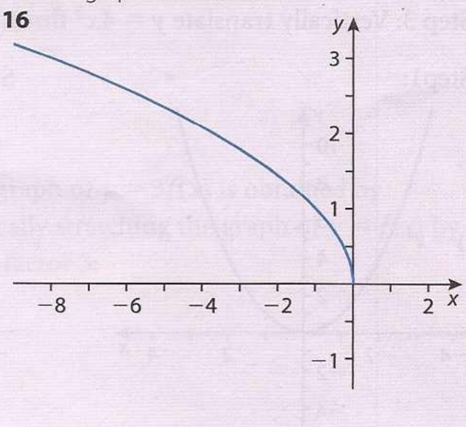

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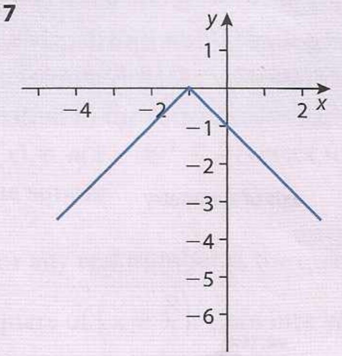
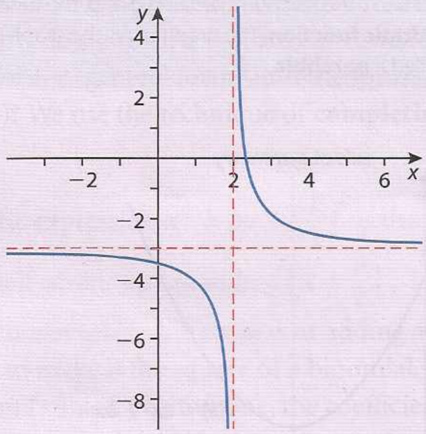
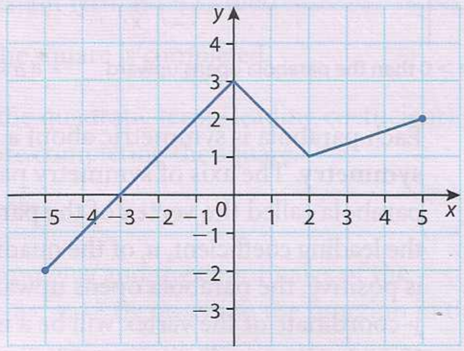


18 Vertical and horizontal asymptotes shown:



19 The graph of f is given. Sketch the graphs of the following functions.

a) $y = f(x) - 3$
b) $y = f(x - 3)$
c) $y = 2f(x)$
d) $y = f(2x)$
e) $y = -f(x)$
f) $y = f(-x)$
g) $y = 2f(x) + 4$



In questions 20–23, specify a sequence of transformations to perform on the graph of $y = x^2$ to obtain the graph of the given function.

20 $g: x \mapsto (x-3)^2 + 5$ **21** $h: x \mapsto -x^2 + 2$

22 $p: x \mapsto \frac{1}{2}(x+4)^2$ **23** $f: x \mapsto [3(x-1)]^2 - 6$

In questions 24–26, a) express the quadratic function in the form $f(x) = a(x-h)^2 + k$, and b) state the coordinates of the vertex of the parabola with equation $y = f(x)$.

24 $f(x) = x^2 + 6x + 2$

25 $f(x) = x^2 - 2x + 4$

26 $f(x) = 4x^2 - 4x - 1$

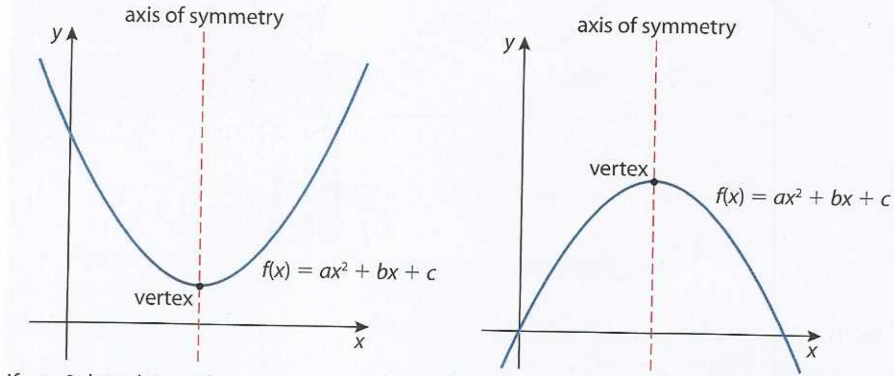
2.5 Quadratic functions

A **linear function** is a polynomial function of degree one that can be written in the general form $f(x) = ax + b$, where $a \neq 0$. The **degree** of a polynomial written in terms of x refers to the largest exponent for x in any terms of the polynomial. In this section, we will consider **quadratic functions** that are second degree polynomial functions, often written in the general form $f(x) = ax^2 + bx + c$. Examples of quadratic functions, such as $f(x) = x^2 + 2$ (where $a = 1$, $b = 0$ and $c = 2$) and $f(x) = x^2 - 4x$ (where $a = 1$, $b = -4$ and $c = 0$), appeared earlier in this chapter.

The word *quadratic* comes from the Latin word *quadratus* that means four-sided, to make square, or simply a square. *Numerus quadratus* means a square number. Before modern algebraic notation was developed in the 17th and 18th centuries, the geometric figure of a square was used to indicate a number multiplying itself. Hence, raising a number to the power of two (in modern notation) is commonly referred to as the operation of squaring. *Quadratic* then came to be associated with a polynomial of degree two rather than being associated with the number four, as the prefix *quad* often indicates (e.g. quadruple).

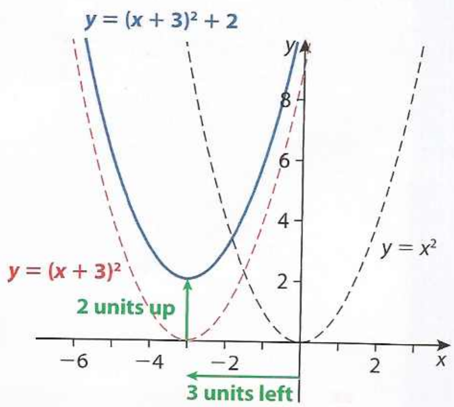
Definition of a quadratic function
If a, b and c are real numbers, and $a \neq 0$, the function $f(x) = ax^2 + bx + c$ is a **quadratic function**. The graph of f is the graph of the equation $y = ax^2 + bx + c$ and is called a **parabola**.

Figure 2.32



Each parabola is symmetric about a vertical line called its **axis of symmetry.** The axis of symmetry passes through a point on the parabola called the **vertex** of the parabola, as shown in Figure 2.32. If the leading coefficient, a, of the quadratic function $f(x) = ax^2 + bx + c$ is positive, the parabola opens upward (concave up) – and the y-coordinate of the vertex will be a **minimum value** for the function. If the leading coefficient, a, of $f(x) = ax^2 + bx + c$ is negative, the parabola opens downward (concave down) – and the y-coordinate of the vertex will be a **maximum value** for the function.

Figure 2.33



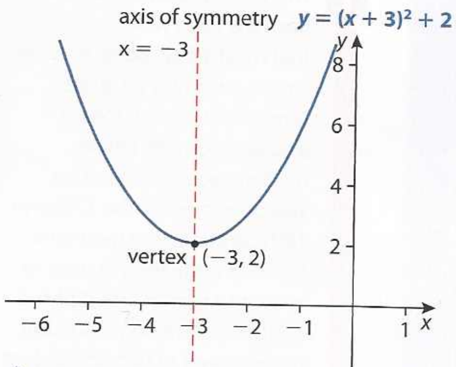


Figure 2.34

The graph of $f(x) = a(x - h)^2 + k$

From the previous section, we know that the graph of the equation $y = (x + 3)^2 + 2$ can be obtained by translating $y = x^2$ three units to the left and two units up. Being familiar with the shape and position of the graph of $y = x^2$, and knowing the two translations that transform $y = x^2$ to $y = (x + 3)^2 + 2$, we can easily visualize and/or sketch the graph of $y = (x + 3)^2 + 2$ (see Figure 2.33). We can also determine the axis of symmetry and the vertex of the graph. Figure 2.34 shows that the graph of $y = (x + 3)^2 + 2$ has an axis of symmetry of $x = -3$ and a vertex at $(-3, 2)$. The equation $y = (x + 3)^2 + 2$ can also be written as $y = x^2 + 6x + 11$. Because we can easily identify the vertex of the parabola when the equation is written as $y = (x + 3)^2 + 2$, we often refer to this as the **vertex form** of the quadratic equation, and $y = x^2 + 6x + 11$ as the **general form**.

• **Hint:** $f(x) = a(x - h)^2 + k$ is sometimes referred to as the **standard form** of a quadratic function.

Vertex form of a quadratic function
If a quadratic function is written in the form $f(x) = a(x - h)^2 + k$, with $a \neq 0$, the graph of f has an axis of symmetry of $x = h$ and a vertex at (h, k).

Completing the square

For visualizing and sketching purposes, it is helpful to have a quadratic function written in vertex form. How do we rewrite a quadratic function written in the form $f(x) = ax^2 + bx + c$ (general form) into the form $f(x) = a(x - h)^2 + k$ (vertex form)? We use the technique of **completing the square.**

For any real number p, the quadratic expression $x^2 + px + \left(\frac{p}{2}\right)^2$ is the square of $\left(x + \frac{p}{2}\right)$. Convince yourself of this by expanding $\left(x + \frac{p}{2}\right)^2$. The technique of *completing the square* is essentially the process of adding a constant to a quadratic expression to make it the square of a binomial. If the coefficient of the quadratic term (x^2) is a positive one, the coefficient of the linear term is p, and the constant term is $\left(\frac{p}{2}\right)^2$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = \left(x + \frac{p}{2}\right)^2$ and the square is completed.

Remember that the coefficient of the quadratic term (leading coefficient) must be equal to positive one before completing the square.

Example 23

Find the equation of the axis of symmetry and the coordinates of the vertex of the graph of $f(x) = x^2 - 8x + 18$ by rewriting the function in the form $f(x) = a(x - h)^2 + k$.

Solution

To complete the square and get the quadratic expression $x^2 - 8x + 18$ in the form $x^2 + px + \left(\frac{p}{2}\right)^2$, the constant term needs to be $\left(\frac{-8}{2}\right)^2 = 16$. We need to add 16, but also subtract 16, so that we are adding zero overall and, hence, not changing the original expression.

$f(x) = x^2 - 8x + 16 - 16 + 18$	actually adding zero $(-16 + 16)$ to the right side
$f(x) = x^2 - 8x + 16 + 2$	$x^2 - 8x + 16$ fits the pattern $x^2 + px + \left(\frac{p}{2}\right)^2$ with $p = -8$
$f(x) = (x - 4)^2 + 2$	$x^2 - 8x + 16 = (x - 4)^2$

The axis of symmetry of the graph of f is the vertical line $x = 4$ and the vertex is at (4, 2). See Figure 2.35.

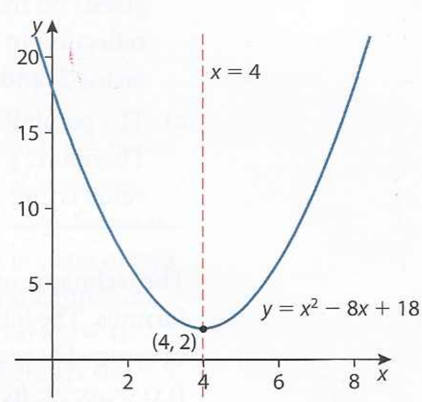


Figure 2.35

Example 24

For the function $g: x \mapsto -2x^2 - 12x + 7$,

a) find the axis of symmetry and the vertex of the graph

b) indicate the transformations that can be applied to $y = x^2$ to obtain the graph

c) find the minimum or maximum value.

Solution

a) $g: x \mapsto -2\left(x^2 + 6x - \frac{7}{2}\right)$ — factorize so that the coefficient of the quadratic term is $+1$

$g: x \mapsto -2\left(x^2 + 6x + 9 - 9 - \frac{7}{2}\right)$ — $p = 6 \Rightarrow \left(\frac{p}{2}\right)^2 = 9$; hence, add $+9 - 9$ (zero)

$g: x \mapsto -2\left[(x+3)^2 - \frac{18}{2} - \frac{7}{2}\right]$ — $x^2 + 6x + 9 = (x+3)^2$

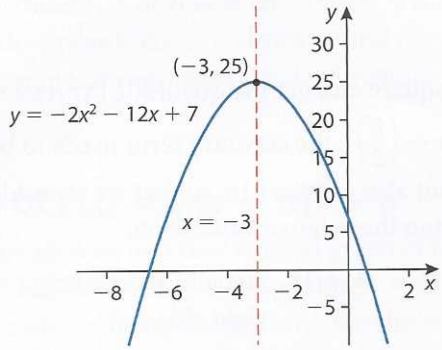
$g: x \mapsto -2\left[(x+3)^2 - \frac{25}{2}\right]$

$g: x \mapsto -2(x+3)^2 + 25$ — multiply through by -2 to remove outer brackets

$g: x \mapsto -2(x-(-3))^2 + 25$ — express in vertex form: $g: x \mapsto a(x-h)^2 + k$

The axis of symmetry of the graph of g is the vertical line $x = -3$ and the vertex is at $(-3, 25)$. See Figure 2.36.

Figure 2.36



b) Since $g: x \mapsto -2x^2 - 12x + 7 = -2(x+3)^2 + 25$, the graph of g can be obtained by applying the following transformations (in the order given) on the graph of $y = x^2$: horizontal translation of 3 units left; reflection in the x-axis (parabola opening down); vertical stretch of factor 2; and a vertical translation of 25 units up.

c) The parabola opens down because the leading coefficient is negative. Therefore, g has a maximum and no minimum value. The maximum value is 25 (y-coordinate of vertex) at $x = -3$.

The technique of completing the square can be used to derive the quadratic formula. The following example derives a general expression for the axis of symmetry and vertex of a quadratic function in the general form $f(x) = ax^2 + bx + c$ by completing the square.

Example 25

Find the axis of symmetry and the vertex for the general quadratic function $f(x) = ax^2 + bx + c$.

Solution

$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$ — factorize so that the coefficient of the x^2 term is $+1$

$f(x) = a\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right]$ — $p = \frac{b}{a} \Rightarrow \left(\frac{p}{2}\right)^2 = \left(\frac{b}{2a}\right)^2$

$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right]$ — $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$

$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$ — multiply through by a

$f(x) = a\left(x - \left(-\frac{b}{2a}\right)\right)^2 + c - \frac{b^2}{4a}$ — express in vertex form: $f(x) = a(x - h)^2 + k$

This result leads to the following generalization.

Symmetry and vertex of $f(x) = ax^2 + bx + c$

For the graph of the quadratic function $f(x) = ax^2 + bx + c$, the axis of symmetry is the vertical line with the equation $x = -\frac{b}{2a}$ and the vertex has coordinates $\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right)$.

Check the results for Example 24 using the formulae for the axis of symmetry and vertex. For the function $g: x \mapsto -2x^2 - 12x + 7$:

$x = -\frac{b}{2a} = -\frac{-12}{2(-2)} = -3 \Rightarrow$ axis of symmetry is the vertical line $x = -3$

$c - \frac{b^2}{4a} = 7 - \frac{(-12)^2}{4(-2)} = \frac{56}{8} + \frac{144}{8} = 25 \Rightarrow$ vertex has coordinates $(-3, 25)$

These results agree with the results from Example 24.

Zeros of a quadratic function

A specific value for x is a **zero** (or **root**) of a quadratic function $f(x) = ax^2 + bx + c$ if it is a solution to the equation $ax^2 + bx + c = 0$. For this course, we are only concerned with values of x that are real numbers. The x-coordinate of any point(s) where f crosses the x-axis (y-coordinate is zero) is a zero of the function. A quadratic function can have no, one or two real zeros as Table 2.3 illustrates. Finding the zeros of a quadratic function requires you to solve quadratic equations of the form $ax^2 + bx + c = 0$. Although $a \neq 0$, it is possible for b or c to be equal to zero. There are five general methods for solving quadratic equations as outlined in Table 2.3.

Square root **Examples**	If $a^2 = c$ and $c > 0$, then $a = \pm\sqrt{c}$. $x^2 - 25 = 0$ $\quad$ $(x+2)^2 = 15$ $x^2 = 25$ $\quad$ $x + 2 = \pm\sqrt{15}$ $x = \pm\sqrt{5}$ $\quad$ $x = -2 \pm\sqrt{15}$
Factorizing **Examples**	If $ab = 0$, then $a = 0$ or $b = 0$. $x^2 + 3x - 10 = 0$ $\quad$ $x^2 - 7x = 0$ $(x+5)(x-2) = 0$ $\quad$ $x(x-7) = 0$ $x = -5$ or $x = 2$ $\quad$ $x = 0$ or $x = 7$
Completing the square **Example**	If $x^2 + px + q = 0$, then $x^2 + px + \left(\frac{p}{2}\right)^2 = -q + \left(\frac{p}{2}\right)^2$ which leads to $\left(x + \frac{p}{2}\right)^2 = -q + \frac{p^2}{4} \ldots$ and then the square root of both sides (as above). $x^2 - 8x + 5 = 0$ $x^2 - 8x + 16 = -5 + 16$ $(x-4)^2 = 11$ $x - 4 = \pm\sqrt{11}$ $x = 4 \pm \sqrt{11}$
Quadratic formula **Example**	If $ax^2 + bx + c = 0$, then $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. $2x^2 - 3x - 4 = 0$ $x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-4)}}{2(2)}$ $x = \frac{3 \pm \sqrt{41}}{4}$
Graphing **Example**	Graph the equation $y = ax^2 + bx + c$ on your GDC. Use the calculating features of your GDC to determine the x-coordinates of the point(s) where the parabola intersects the x-axis. $2x^2 - 5x - 7 = 0$ $\quad$ GDC calculations reveal that the zeros are at $x = \frac{7}{2}$ and $x = -1$

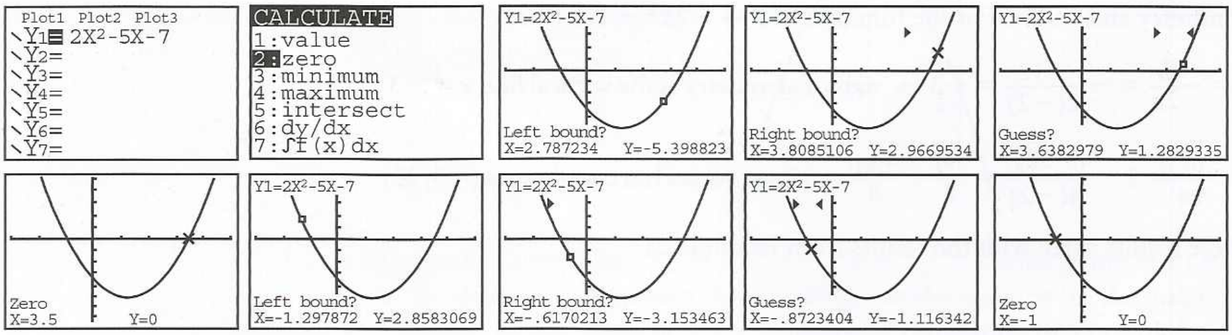


Table 2.3 Methods for solving quadratic equations.

The quadratic formula and the discriminant

The expression $b^2 - 4ac$ in the quadratic formula has special significance because you need to take the positive and negative square root of $b^2 - 4ac$ when using the quadratic formula. Hence, whether $b^2 - 4ac$ (often labelled Δ; read 'delta') is positive, negative or zero will determine the number of real solutions for the quadratic equation $ax^2 + bx + c = 0$, and, consequently, also the number of times the graph of $f(x) = ax^2 + bx + c$ intersects the x-axis ($y = 0$).

For the quadratic function $f(x) = ax^2 + bx + c, a \neq 0$:

If $\Delta = b^2 - 4ac > 0$, f has two distinct real solutions, and the graph of f intersects the x-axis twice.

If $\Delta = b^2 - 4ac = 0$, f has one real solution (a double root), and the graph of f intersects the x-axis once (i.e. it is tangent to the x-axis).

If $\Delta = b^2 - 4ac < 0$, f has no real solutions, and the graph of f does not intersect the x-axis.

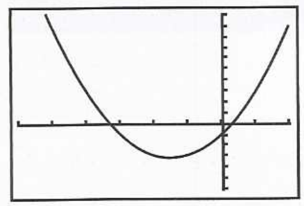
Example 26

Use the discriminant to determine how many real solutions each equation has. Visually confirm the result by graphing the corresponding quadratic function for each equation on your GDC.

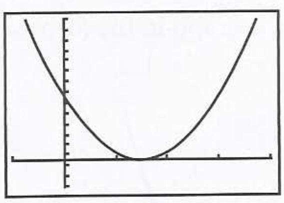
a) $x^2 + 3x - 1 = 0$ b) $4x^2 - 12x + 9 = 0$ c) $2x^2 - 5x + 6 = 0$

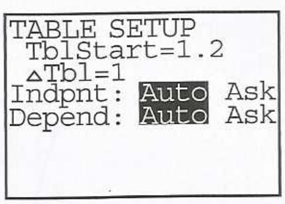
Solution

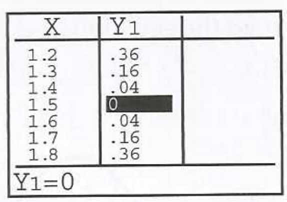
a) The discriminant is $\Delta = 3^2 - 4(1)(-1) = 13 > 0$. Therefore, the equation has two distinct real zeros. This result is confirmed by the graph of the quadratic function $y = x^2 + 3x - 1$ which clearly shows it intersecting the x-axis twice as shown in GDC image on the right.



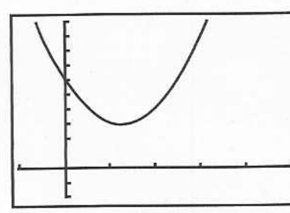
b) The discriminant is $\Delta = (-12)^2 - 4(4)(9) = 0$. Therefore, the equation has one real zero. The graph on the GDC of $y = 4x^2 - 12x + 9$ appears to intersect the x-axis at only one point. We can be more confident with this conclusion by investigating further – for example, tracing or looking at a table of values on the GDC as shown in GDC images below.







c) The discriminant is $\Delta = (-5)^2 - 4(2)(6) = -23 < 0$. Therefore, the equation has no real zeros. This result is confirmed by the graph of the quadratic function $y = 2x^2 - 5x + 6$ which clearly shows that the graph does not intersect the x-axis as shown in GDC image on the right.



Example 27

For $4x^2 + 4kx + 9 = 0$, determine the value(s) of k so that the equation has a) one real zero, b) two distinct real zeros, and c) no real zeros.

Solution

a) For one real zero: $\Delta = (4k)^2 - 4(4)(9) = 0 \Rightarrow 16k^2 - 144 = 0$
$\Rightarrow 16k^2 = 144 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

b) For two distinct real zeros: $\Delta = (4k)^2 - 4(4)(9) > 0$
$\Rightarrow 16k^2 > 144 \Rightarrow k^2 > 9 \Rightarrow k < -3$ or $k > 3$

c) For no real zeros: $\Delta = (4k)^2 - 4(4)(9) < 0 \Rightarrow 16k^2 < 144$
$\Rightarrow k^2 < 9 \Rightarrow k > -3$ and $k < 3 \Rightarrow -3 < k < 3$

The graph of $f(x) = a(x - p)(x - q)$

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$ then we can easily identify the x-intercepts of the graph of f. Consider that $f(p) = a(p - p)(p - q) = a(0)(p - q) = 0$ and that $f(q) = a(q - p)(q - q) = a(q - p)(0) = 0$. Therefore, the quadratic function $f(x) = a(x - p)(x - q)$ will intersect the x-axis at the points $(p, 0)$ and $(q, 0)$. We need to factorize in order to rewrite a quadratic function in the form $f(x) = ax^2 + bx + c$ to the form $f(x) = a(x - p)(x - q)$. Hence, $f(x) = a(x - p)(x - q)$ can be referred to as the **factorized** form of a quadratic function. Recalling the symmetric nature of a parabola, it is clear that the x-intercepts $(p, 0)$ and $(q, 0)$ will be equidistant from the axis of symmetry (see Figure 2.37). As a result, the equation of the axis of symmetry and the x-coordinate of the vertex of the parabola can be found from finding the average of p and q.

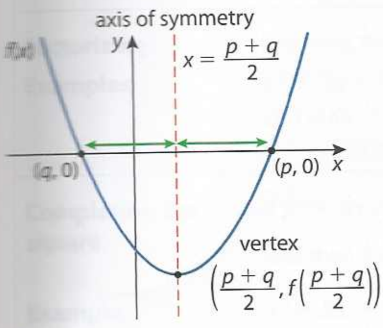


Figure 2.37

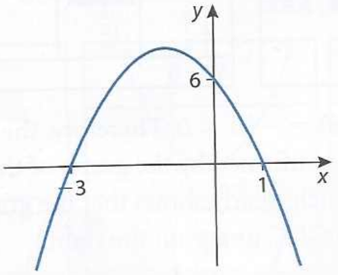
Factorized form of a quadratic function

If a quadratic function is written in the form $f(x) = a(x - p)(x - q)$, with $a \neq 0$, the graph of f has x-intercepts at $(p, 0)$ and $(q, 0)$, an axis of symmetry with equation $x = \frac{p+q}{2}$, and a vertex at $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$.

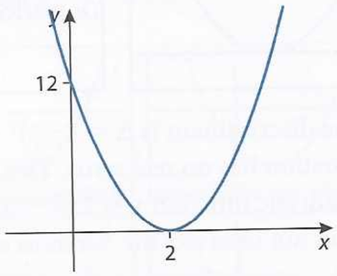
Example 28

Find the equation of each quadratic function from the graph in the form $f(x) = a(x - p)(x - q)$ and also in the form $f(x) = ax^2 + bx + c$.

a)



b)



Solution

a) Since the x-intercepts are -3 and 1 then $y = a(x + 3)(x - 1)$. The y-intercept is 6, so when $x = 0$, $y = 6$. Hence, $6 = a(0 + 3)(0 - 1) = -3a \Rightarrow a = -2$ ($a < 0$ agrees with the fact that the parabola is opening down). The function is $f(x) = -2(x + 3)(x - 1)$, and expanding to remove brackets reveals that the function can also be written as $f(x) = -2x^2 - 4x + 6$.

b) The function has one x-intercept at 2 (double root), so $p = q = 2$ and $y = a(x - 2)(x - 2) = a(x - 2)^2$. The y-intercept is 12, so when $x = 0$, $y = 12$. Hence, $12 = a(0 - 2)^2 = 4a \Rightarrow a = 3$ ($a > 0$ agrees with the parabola opening up). The function is $f(x) = 3(x - 2)^2$. Expanding reveals that the function can also be written as $f(x) = 3x^2 - 12x + 12$.

Example 29

The graph of a quadratic function intersects the x-axis at the points $(-6, 0)$ and $(-2, 0)$ and also passes through the point $(2, 16)$. a) Write the function in the form $f(x) = a(x - p)(x - q)$. b) Find the vertex of the parabola. c) Write the function in the form $f(x) = a(x - h)^2 + k$.

Solution

a) The x-intercepts of -6 and -2 gives $f(x) = a(x + 6)(x + 2)$. Since f passes through $(2, 16)$, then $f(2) = 16 \Rightarrow f(2) = a(2 + 6)(2 + 2) = 16$ $\Rightarrow 32a = 16 \Rightarrow a = \frac{1}{2}$. Therefore, $f(x) = \frac{1}{2}(x + 6)(x + 2)$.

b) The x-coordinate of the vertex is the average of the x-intercepts. $x = \dfrac{-6 - 2}{2} = -4$, so the y-coordinate of the vertex is $y = f(-4) = \frac{1}{2}(-4 + 6)(-4 + 2) = -2$. Hence, the vertex is $(-4, -2)$.

c) In vertex form, the quadratic function is $f(x) = \frac{1}{2}(x + 4)^2 - 2$.

Exercise 2.5

For each of the quadratic functions f in questions 1–5, find the following:

a) the axis of symmetry and the vertex, by algebraic methods
b) the transformation(s) that can be applied to $y = x^2$ to obtain the graph of $y = f(x)$
c) the minimum or maximum value of f.

Check your results using your GDC.

1 $f: x \mapsto x^2 - 10x + 32$

2 $f: x \mapsto x^2 + 6x + 8$

3 $f: x \mapsto -2x^2 - 4x + 10$

4 $f: x \mapsto 4x^2 - 4x + 9$

5 $f: x \mapsto \frac{1}{2}x^2 + 7x + 26$

In questions 6–13, solve the quadratic equation using factorization.

6 $x^2 + 2x - 8 = 0$

7 $x^2 = 3x + 10$

8 $6x^2 - 9x = 0$

9 $6 + 5x = x^2$

10 $x^2 + 9 = 6x$

11 $3x^2 + 11x - 4 = 0$

12 $3x^2 + 18 = 15x$

13 $9x - 2 = 4x^2$

In questions 14–19, use the method of completing the square to solve the quadratic equation.

14 $x^2 + 4x - 3 = 0$

15 $x^2 - 4x - 5 = 0$

16 $x^2 - 2x + 3 = 0$

17 $2x^2 + 16x + 6 = 0$

18 $x^2 + 2x - 8 = 0$

19 $-2x^2 + 4x + 9 = 0$

20 Let $f(x) = x^2 - 4x - 1$. a) Use the quadratic formula to find the zeros of the function. b) Use the zeros to find the equation for the axis of symmetry of the parabola. c) Find the minimum or maximum value of f.

In questions 21–24, determine the number of real solutions to each equation.

21 $x^2 + 3x + 2 = 0$

22 $2x^2 - 3x + 2 = 0$

23 $x^2 - 1 = 0$

24 $2x^2 - \frac{9}{4}x + 1 = 0$

25 Find the value(s) of p for which the equation $2x^2 + px + 1 = 0$ has one real solution.

26 Find the value(s) of k for which the equation $x^2 + 4x + k = 0$ has two distinct real solutions.

27 The equation $x^2 - 4kx + 4 = 0$ has two distinct real solutions. Find the set of all possible values of k.

28 Find all possible values of m so that the graph of the function $g: x \mapsto mx^2 + 6x + m$ does not touch the x-axis.

Practice questions

1 Let $f: x \mapsto \sqrt{x-3}$ and $g: x \mapsto x^2 + 2x$. The function $(f \circ g)(x)$ is defined for all $x \in \mathbb{R}$ **except** for the interval $]a, b[$.

a) Calculate the values of a and b.

b) Find the range of $f \circ g$.

2 Two functions g and h are defined as $g(x) = 2x - 7$ and $h(x) = 3(2 - x)$.
Find: **a)** $g^{-1}(3)$
b) $(h \circ g)(6)$

3 Consider the functions $f(x) = 5x - 2$ and $g(x) = \frac{4 - x}{3}$.

a) Find g^{-1}.

b) Solve the equation $(f \circ g^{-1})(x) = 8$.

4 The functions g and h are defined by $g: x \mapsto x - 3$ and $h: x \mapsto 2x$.

a) Find an expression for $(g \circ h)(x)$.

b) Show that $g^{-1}(14) + h^{-1}(14) = 24$.

5 The function f is defined by $f(x) = x^2 + 8x + 11$, for $x \geqslant -4$.

a) Write $f(x)$ in the form $(x - h)^2 + k$.

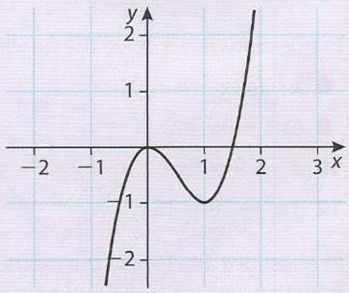
b) Find the inverse function f^{-1}.

c) State the domain of f^{-1}.

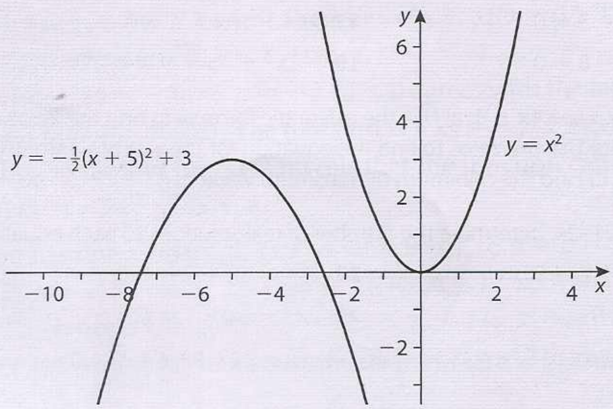
6 The diagram right shows the graph of $y = f(x)$. It has maximum and minimum points at $(0, 0)$ and $(1, -1)$, respectively.

a) Copy the diagram and, on the same diagram, draw the graph of $y = f(x + 1) - \frac{1}{2}$.

b) What are the coordinates of the minimum and maximum points of $y = f(x + 1) - \frac{1}{2}$?



7 The diagram shows parts of the graphs of $y = x^2$ and $y = -\frac{1}{2}(x + 5)^2 + 3$.



The graph of $y = x^2$ may be transformed into the graph of $y = -\frac{1}{2}(x + 5)^2 + 3$ by these transformations.

A reflection in the line $y = 0$, followed by
a vertical stretch by scale factor k, followed by
a horizontal translation of p units, followed by
a vertical translation of q units.

Write down the value of

a) k **b)** p **c)** q.

8 The function f is defined by $f(x) = \frac{4}{\sqrt{16 - x^2}}$, for $-4 < x < 4$.

a) Without using a GDC, sketch the graph of f.

b) Write down the equation of each vertical asymptote.

c) Write down the range of the function f.

9 Let $g: x \mapsto \frac{1}{x}, x \neq 0$.

a) Without using a GDC, sketch the graph of g.

The graph of g is transformed to the graph of h by a translation of 4 units to the left and 2 units down.

b) Find an expression for the function h.

c) **(i)** Find the x- and y-intercepts of h.
(ii) Write down the equations of the asymptotes of h.
(iii) Sketch the graph of h.

10 Consider $f(x) = \sqrt{x + 3}$.

a) Find:
(i) $f(8)$
(ii) $f(46)$
(iii) $f(-3)$

b) Find the values of x for which f is undefined.

c) Let $g: x \mapsto x^2 - 5$. Find $(g \circ f)(x)$.

11 Let $g(x) = \frac{x - 8}{2}$ and $h(x) = x^2 - 1$.

a) Find $g^{-1}(-2)$.

b) Find an expression for $(g^{-1} \circ h)(x)$.

c) Solve $(g^{-1} \circ h)(x) = 22$.

12 Given the functions $f: x \mapsto 3x - 1$ and $g: x \mapsto \frac{4}{x}$, find the following:

a) f^{-1} **b)** $f \circ g$ **c)** $(f \circ g)^{-1}$ **d)** $g \circ g$

13 The quadratic function f is defined by $f(x) = 2x^2 + 8x + 17$.

a) Write f in the form $f(x) = 2(x - h)^2 + k$.

b) The graph of f is translated 5 units in the positive x-direction and 2 units in the positive y-direction. Find the function g for the translated graph, giving your answer in the form $g(x) = 2(x - h)^2 + k$.

14 Let $g(x) = 3x^2 - 6x - 4$.

a) Express $g(x)$ in the form $g(x) = 3(x - h)^2 + k$.

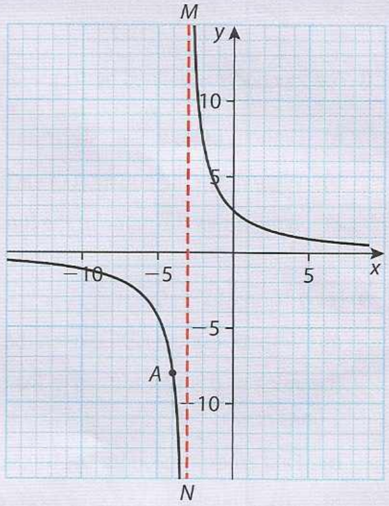
b) Write down the vertex of the graph of g.

c) Write down the equation of the axis of symmetry of the graph of g.

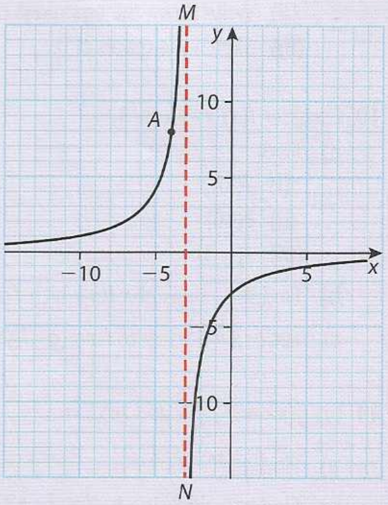
d) Find the y-intercept of the graph of g.

e) The x-intercepts of g can be written as $\frac{p \pm q}{r}$, where $p, q, r \in \mathbb{Z}$. Find the value of p, q and r.

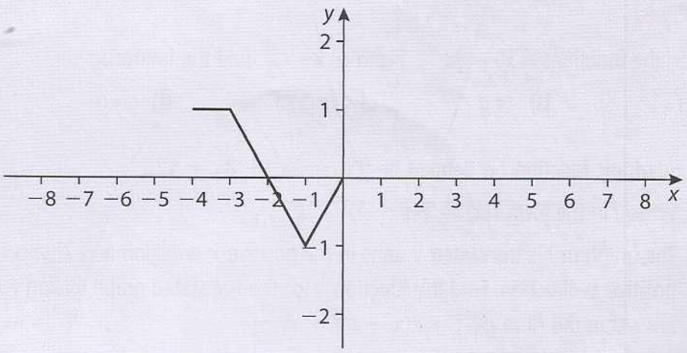
15 a) The diagram shows part of the graph of the function $h(x) = \frac{a}{x - b}$. The curve passes through the point $A(-4, -8)$. The vertical line (MN) is an asymptote. Find the value of: **(i)** a **(ii)** b.



b) The graph of $h(x)$ is transformed as shown in the diagram right. The point A is transformed to $A'(-4, 8)$. Give a full geometric description of the transformation.



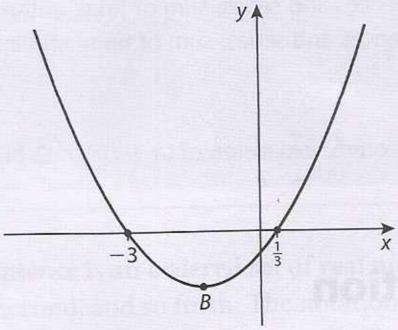
16 The graph of $y = f(x)$ is shown in the diagram.



a) Make two copies of the coordinate system as shown in the diagram but without the graph of $y = f(x)$. On the first diagram sketch a graph of $y = 2f(x)$, and on the second diagram sketch a graph of $y = f(x - 4)$.

b) The point $A(-3, 1)$ is on the graph of $y = f(x)$. The point A' is the corresponding point on the graph of $y = -f(x) - 1$. Find the coordinates of A'.

17 The diagram represents the graph of the function $f(x) = (x - p)(x - q)$.



a) Write down the values of p and q.

b) The function has a minimum value at the point B. Find the x-coordinate of B.

c) Write the expression for $f(x)$ in the form $ax^2 + bx + c$.

18 The diagram shows the parabola $y = (5 + x)(2 - x)$. The points A and C are the x-intercepts and the point B is the maximum point. Find the coordinates of A, B and C.