3 Sequences and Series

Assessment statements

1.1 Arithmetic sequences and series; sum of finite arithmetic sequences; geometric sequences and series; sum of finite and infinite geometric series.
Sigma notation.

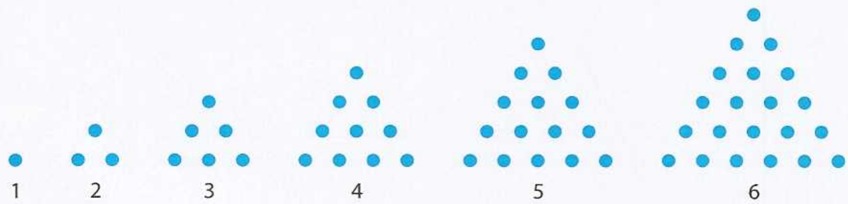
1.3 The binomial theorem: expansion of $(a + b)^n$, $n \in \mathbb{N}$.

Introduction

The heights of consecutive bounds of a ball, compound interest and Fibonacci numbers are only a few of the applications of sequences and series that you have seen in previous courses. In this chapter, you will review these concepts, consolidate your understanding and take them one step further.

3.1 Sequences

Take the following pattern as an example:



The first figure represents 1 dot, the second represents 3 dots, etc. This pattern can also be described differently. For example, in function notation:

$$f(1) = 1, f(2) = 3, f(3) = 6, \text{ etc., where the domain is } \mathbb{Z}^+$$

Here are some more examples of sequences:

1 6, 12, 18, 24, 30

2 $3, 9, 27, \ldots, 3^k, \ldots$

3 $\left\{\frac{1}{i^2}; i = 1, 2, 3, \ldots, 10\right\}$

4 $\{b_1, b_2, \ldots, b_n, \ldots\}$, sometimes used with an abbreviation $\{b_n\}$

The first and third sequences are **finite** and the second and fourth are **infinite**. Notice that, in the second and third sequences, we were able to define a rule that yields the nth number in the sequence (called the nth term) as a function of n, the term's number. In this sense, a sequence is a **function** that assigns a **unique** number (a_n) to each positive integer n.

Example 1

Find the first five terms and the 50th term of the sequence $\{b_n\}$ such that $b_n = 2 - \frac{1}{n^2}$.

Solution

Since we know an *explicit* expression for the nth term as a *function* of its number n, we only need to find the value of that function for the required terms:

$b_1 = 2 - \frac{1}{1^2} = 1$; $b_2 = 2 - \frac{1}{2^2} = 1\frac{3}{4}$; $b_3 = 2 - \frac{1}{3^2} = 1\frac{8}{9}$; $b_4 = 2 - \frac{1}{4^2} = 1\frac{15}{16}$;

$b_5 = 2 - \frac{1}{5^2} = 1\frac{24}{25}$; and $b_{50} = 2 - \frac{1}{50^2} = 1\frac{2499}{2500}$.

So, informally, **a sequence is an ordered set of real numbers.** That is, there is a first number, a second, and so forth. The notation used for such sets is shown above. The way we defined the function in Example 1 is called the **explicit** definition of a sequence. There are other ways to define sequences, one of which is the **recursive** definition. The following example will show you how this is used.

Example 2

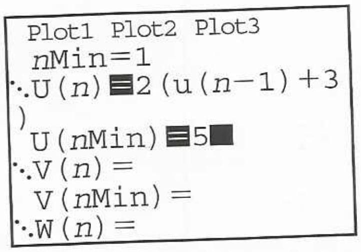
Find the first five terms and the 20th term of the sequence $\{b_n\}$ such that $b_1 = 5$ and $b_n = 2(b_{n-1} + 3)$.

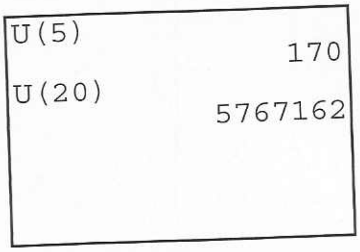
- **Hint:** This can easily be done using a GDC.

Solution

The defining formula for this sequence is recursive. It allows us to find the nth term b_n if we know the preceding term b_{n-1}. Thus, we can find the second term from the first, the third from the second, and so on. Since we know the first term, $b_1 = 5$, we can calculate the rest:

$$b_2 = 2(b_1 + 3) = 2(5 + 3) = 16$$
$$b_3 = 2(b_2 + 3) = 2(16 + 3) = 38$$
$$b_4 = 2(b_3 + 3) = 2(38 + 3) = 82$$
$$b_5 = 2(b_4 + 3) = 2(82 + 3) = 170$$





Thus, the first five terms of this sequence are 5, 16, 38, 82, 170. However, to find the 20th term, we must first find all 19 preceding terms. This is one of the drawbacks of the recursive definition, unless we can change the definition into explicit form.

However, you need to understand that not all sequences have formulae, either recursive or explicit. Some sequences are given only by listing their terms. Among the many kinds of sequences that there are, two types are of interest to us: arithmetic and geometric sequences.

Exercise 3.1

Find the first five terms and the 50th term of each infinite sequence defined in questions 1–8.

1 $a_n = 2n - 3$

2 $b_n = 2 \times 3^{n-1}$

3 $u_n = (-1)^{n-1}\dfrac{2n}{n^2 + 2}$

4 $a_n = n^{n-1}$

5 $a_n = 2a_{n-1} + 5$ and $a_1 = 3$

6 $u_{n+1} = \dfrac{3}{2u_n + 1}$ and $u_1 = 0$

7 $b_n = 3 \cdot b_{n-1}$ and $b_1 = 2$

8 $a_n = a_{n-1} + 2$ and $a_1 = -1$

3.2 Arithmetic sequences

Examine the following sequences and the most likely recursive formula for each of them.

7, 14, 21, 28, 35, 42, … $a_1 = 7$ and $a_n = a_{n-1} + 7$, for $n > 1$

2, 11, 20, 29, 38, 47, … $a_1 = 2$ and $a_n = a_{n-1} + 9$, for $n > 1$

48, 39, 30, 21, 12, 3, −6, … $a_1 = 48$ and $a_n = a_{n-1} - 9$, for $n > 1$

Note that in each case above, every term is formed by adding a constant number to the preceding term. Sequences formed in this manner are called **arithmetic sequences.**

Definition of an arithmetic sequence

A sequence $a_1, a_2, a_3, \ldots$ is an **arithmetic sequence** if there is a constant d for which

$$a_n = a_{n-1} + d$$

for all integers $n > 1$. d is called the **common difference** of the sequence, and $d = a_n - a_{n-1}$ for all integers $n > 1$.

So, for the sequences above, 7 is the common difference for the first, 9 is the common difference for the second and −9 is the common difference for the third.

This description gives us the recursive definition of the arithmetic sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly will enable you to see the expression we are seeking:

$$a_2 = a_1 + d;\ a_3 = a_2 + d = a_1 + d + d = a_1 + 2d;$$
$$a_4 = a_3 + d = a_1 + 2d + d = a_1 + 3d;\ \ldots$$

So, as you see, you can get to the nth term by adding d to a_1, $(n - 1)$ times, and therefore:

***n*th term of an arithmetic sequence**

The general (nth) term of an arithmetic sequence, a_n, with first term a_1 and common difference d, may be expressed explicitly as

$$a_n = a_1 + (n - 1)d$$

This result is useful in finding any term of the sequence without knowing all the previous terms.

Note: The arithmetic sequence can be looked at as a linear function as explained in the introduction to this chapter, i.e. for every increase of one unit in n, the value of the term will increase by d units. As the first term is a_1, the point $(1, a_1)$ belongs to this function. The constant increase d can be considered to be the gradient (slope) of this linear model; hence, the nth term, the dependent variable in this case, can be found by using the *point-slope* form of the equation of a line:

$$y - y_1 = m(x - x_1)$$
$$a_n - a_1 = d(n - 1) \Leftrightarrow a_n = a_1 + (n - 1)d$$

This agrees with our definition of an arithmetic sequence.

Example 3

Find the nth and the 50th terms of the sequence 2, 11, 20, 29, 38, 47, …

Solution

This is an arithmetic sequence whose first term is 2 and common difference is 9. Therefore,

$$a_n = a_1 + (n - 1)d = 2 + (n - 1) \times 9 = 9n - 7$$
$$\Rightarrow a_{50} = 9 \times 50 - 7 = 443$$

Example 4

Find the recursive and the explicit forms of the definition of the following sequence, then calculate the value of the 25th term.

13, 8, 3, −2, …

Solution

This is clearly an arithmetic sequence, since we observe that −5 is the common difference.

Recursive definition: $a_1 = 13$

$$a_n = a_{n-1} - 5$$

Explicit definition: $a_n = 13 - 5(n - 1) = 18 - 5n$, and

$$a_{25} = 18 - 5 \times 25 = -107$$

Example 5

Find a definition for the arithmetic sequence whose first term is 5 and fifth term is 11.

Solution

Since the fifth term is given, using the explicit form, we have

$$a_5 = a_1 + (5 - 1)d \Rightarrow 11 = 5 + 4d \Rightarrow d = \frac{3}{2}$$

This leads to the general term,

$a_n = 5 + \frac{3}{2}(n - 1)$, or, equivalently, the recursive form

$$\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + \frac{3}{2}, n > 1 \end{cases}$$

● **Hint:** Definition: In a finite arithmetic sequence $a_1, a_2, a_3, \ldots, a_k$, the terms $a_2, a_3, \ldots, a_{k-1}$ are called **arithmetic means** between a_1 and a_k.

Example 6

Insert four arithmetic means between 3 and 7.

Solution

Since there are four means between 3 and 7, the problem can be reduced to a situation similar to Example 5 by considering the first term to be 3 and the sixth term to be 7. The rest is left as an exercise for you!

Exercise 3.2

1 Insert four arithmetic means between 3 and 7.

2 Say whether each given sequence is an arithmetic sequence. If yes, find the common difference and the 50th term; if not, say why not.

a) $a_n = 2n - 3$ b) $b_n = n + 2$

c) $c_n = c_{n-1} + 2$, and $c_1 = -1$ d) $u_n = 3u_{n-1} + 2$

e) $2, 5, 7, 12, 19, \ldots$ f) $2, -5, -12, -19, \ldots$

For each arithmetic sequence in questions 3–8, find:

a) the 8th term

b) an explicit formula for the nth term

c) a recursive formula for the nth term.

3 $-2, 2, 6, 10, \ldots$ **4** $29, 25, 21, 17, \ldots$

5 $-6, 3, 12, 21, \ldots$ **6** $10.07, 9.95, 9.83, 9.71, \ldots$

7 $100, 97, 94, 91, \ldots$ **8** $2, \frac{3}{4}, -\frac{1}{2}, -\frac{7}{4}, \ldots$

9 Find five arithmetic means between 13 and −23.

10 Find three arithmetic means between 299 and 300.

11 In an arithmetic sequence, $a_5 = 6$ and $a_{14} = 42$. Find an explicit formula for the nth term of this sequence.

12 In an arithmetic sequence, $a_3 = -40$ and $a_9 = -18$. Find an explicit formula for the nth term of this sequence.

3.3 Geometric sequences

Examine the following sequences and the most likely recursive formula for each of them.

$7, 14, 28, 56, 112, 224, \ldots$ $a_1 = 7$ and $a_n = a_{n-1} \times 2$, for $n > 1$

$2, 18, 162, 1458, 13\,122, \ldots$ $a_1 = 2$ and $a_n = a_{n-1} \times 9$, for $n > 1$

$48, -24, 12, -6, 3, -1.5, \ldots$ $a_1 = 48$ and $a_n = a_{n-1} \times -0.5$, for $n > 1$

Note that in each case above, every term is formed by multiplying a constant number with the preceding term. Sequences formed in this manner are called **geometric sequences.**

Definition of a geometric sequence

A sequence $a_1, a_2, a_3 \ldots$ is a **geometric sequence** if there is a constant r for which

$$a_n = a_{n-1} \times r$$

for all integers $n > 1$. r is called the **common ratio** of the sequence, and $r = a_n \div a_{n-1}$ for all integers $n > 1$.

Thus, for the sequences above, 2 is the common ratio for the first, 9 is the common ratio for the second and −0.5 is the common ratio for the third.

This description gives us the recursive definition of the geometric sequence. It is possible, however, to find the explicit definition of the sequence.

Applying the recursive definition repeatedly will enable you to see the expression we are seeking:

$$a_2 = a_1 \times r;\ a_3 = a_2 \times r = a_1 \times r \times r = a_1 \times r^2;$$
$$a_4 = a_3 \times r = a_1 \times r^2 \times r = a_1 \times r^3;\ \ldots$$

So, as you see, you can get to the nth term by multiplying a_1 with r, $(n-1)$ times, and therefore:

nth term of geometric sequence

The general (nth) term of a geometric sequence, a_n, with common ratio r and first term a_1, may be expressed explicitly as

$$a_n = a_1 \times r^{(n-1)}$$

This result is useful in finding any term of the sequence without knowing all the previous terms.

Example 7

a) Find the geometric sequence with $a_1 = 2$ and $r = 3$.

b) Describe the sequence 3, −12, 48, −192, 768, …

c) Describe the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$

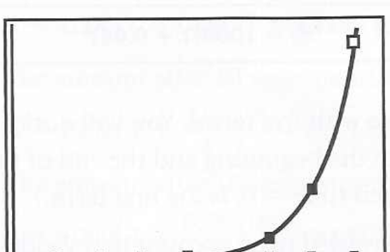
d) Graph the sequence $a_n = \frac{1}{4} \cdot 3^{n-1}$

Solution

a) The geometric sequence is $2, 6, 18, 54, \ldots, 2 \times 3^{n-1}$. Notice that the ratio of a term to the preceding term is 3.

b) This is a geometric sequence with $a_1 = 3$ and $r = -4$. The nth term is $a_n = 3 \times (-4)^{n-1}$. Notice that, when the common ratio is negative, the terms of the sequence alternate in sign.

c) The nth term of this sequence is $a_n = 1 \cdot \left(\frac{1}{2}\right)^{n-1}$. Notice that the ratio of any two consecutive terms is $\frac{1}{2}$. Also, notice that the terms decrease in value.



d) The graph of the geometric sequence is shown on the left. Notice that the points lie on the graph of the function $y = \frac{1}{4} \cdot 3^{x-1}$.

Example 8

At 8:00 a.m., 1000 mg of medicine is administered to a patient. At the end of each hour, the concentration of medicine is 60% of the amount present at the beginning of the hour.

a) What portion of the medicine remains in the patient's body at noon if no additional medication has been given?

b) If a second dosage of 1000 mg is administered at 10:00 a.m., what is the total concentration of the medication in the patient's body at noon?

Solution

a) We use the geometric model, as there is a constant multiple by the end of each hour. Hence, the concentration at the end of any hour after administering the medicine is given by:

$a_n = a_1 \times r^{(n-1)}$, where n is the number of hours

Thus, at noon $n = 5$, and $a_5 = 1000 \times 0.6^{(5-1)} = 129.6$.

b) For the second dosage, the amount of medicine at noon corresponds to $n = 3$, and $a_3 = 1000 \times 0.6^{(3-1)} = 360$.

So, the concentration of medicine is $129.6 + 360 = 489.6$ mg.

See also Section 4.2.

Compound interest

Interest compounded annually

When we borrow money we pay interest, and when we invest money we receive interest. Suppose an amount of €1000 is put into a savings account that bears an annual interest of 6%. How much money will we have in the bank at the end of four years?

It is important to note that the 6% interest is given annually and is added to the savings account, so that in the following year it will also earn interest, and so on.

Time in years	Amount in the account
0	1000
1	$1000 + 1000 \times 0.06 = 1000(1 + 0.06)$
2	$1000(1 + 0.06) + (1000(1 + 0.06)) \times 0.06 = 1000(1 + 0.06)(1 + 0.06) = 1000(1 + 0.06)^2$
3	$1000(1 + 0.06)^2 + (1000(1 + 0.06)^2) \times 0.06 = 1000(1 + 0.06)^2(1 + 0.06) = 1000(1 + 0.06)^3$
4	$1000(1 + 0.06)^3 + (1000(1 + 0.06)^3) \times 0.06 = 1000(1 + 0.06)^3(1 + 0.06) = \mathbf{1000(1 + 0.06)^4}$

Table 3.1 Compound interest.

This appears to be a geometric sequence with five terms. You will notice that the number of terms is five, as both the beginning and the end of the first year are counted. (Initial value, when time = 0, is the first term.)

In general, if a **principal** of P euros is invested in an account that yields an interest rate r (expressed as a decimal) annually, and this interest is added at the end of the year, every year, to the principal, then we can use the geometric sequence formula to calculate the **future value** A, which is accumulated after t years.

If we repeat the steps above, with

$A_0 = P =$ initial amount
$r =$ annual interest rate
$t =$ number of years

it becomes easier to develop the formula:

Time in years	Amount in the account
0	$A_0 = P$
1	$A_1 = P + Pr = P(1 + r)$
2	$A_2 = A_1(1 + r) = P(1 + r)^2$
$\vdots$	
t	$A_t = P(1 + r)^t$

Table 3.2 Compound interest formula.

Notice that since we are counting from 0 to t, we have $t + 1$ terms, and hence using the geometric sequence formula,

$$a_n = a_1 \times r^{(n-1)} \Rightarrow A_t = A_0 \times (1 + r)^t$$

Interest compounded *n* times per year

Suppose that the principal P is invested as before but the interest is paid n times per year. Then $\frac{r}{n}$ is the interest paid every compounding period. Since every year we have n periods, for t years, we have nt periods. The amount A in the account after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example 9

€1000 is invested in an account paying compound interest at a rate of 6%. Calculate the amount of money in the account after 10 years if

a) the compounding is annual
b) the compounding is quarterly
c) the compounding is monthly.

Solution

a) The amount after 10 years is
$A = 1000(1 + 0.06)^{10} = €1790.85.$

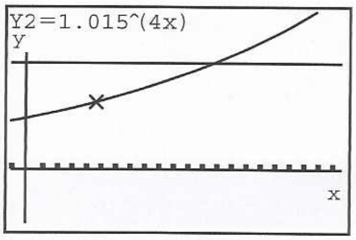
b) The amount after 10 years quarterly compounding is
$A = 1000\left(1 + \frac{0.06}{4}\right)^{40} = €1814.02.$

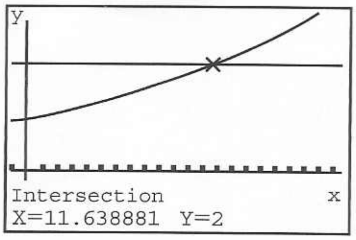
c) The amount after 10 years monthly compounding is
$A = 1000\left(1 + \frac{0.06}{12}\right)^{120} = €1819.40.$

Example 10

You invested €1000 at 6% compounded quarterly. How long will it take this investment to increase to €2000?

Solution





Let $P = 1000$, $r = 0.06$, $n = 4$ and $A = 2000$ in the compound interest formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Then solve for t:

$$2000 = 1000\left(1 + \frac{0.06}{4}\right)^{4t} \Rightarrow 2 = 1.015^{4t}$$

Using a GDC, we can graph the functions $y = 2$ and $y = 1.015^{4t}$ and then find the intersection between their graphs.

As you can see, it will take the €1000 investment 11.64 years to double to €2000. This translates into approximately 47 quarters.

You can check your work to see that this is accurate by using the compound interest formula:

$$A = 1000\left(1 + \frac{0.06}{4}\right)^{47} = €2013.28$$

In the next chapter you will learn how to solve the problem algebraically.

Example 11

You want to invest €1000. What interest rate is required to make this investment grow to €2000 in 10 years if interest is compounded quarterly?

Solution

Let $P = 1000$, $n = 4$, $t = 10$ and $A = 2000$ in the compound interest formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Now solve for r:

$$2000 = 1000\left(1 + \frac{r}{4}\right)^{40} \Rightarrow 2 = \left(1 + \frac{r}{4}\right)^{40} \Rightarrow 1 + \frac{r}{4} = \sqrt[40]{2} \Rightarrow r = 4(\sqrt[40]{2} - 1)$$
$$= 0.0699$$

So, at a rate of 7% compounded quarterly, the €1000 investment will grow to at least €2000 in 10 years.

You can check to see whether your work is accurate by using the compound interest formula:

$$A = 1000\left(1 + \frac{0.07}{4}\right)^{40} = €2001.60$$

Population growth

The same formulae can be applied when dealing with population growth.

Example 12

The city of Baden in Lower Austria grows at an annual rate of 0.35%. The population of Baden in 1981 was 23 140. What is the estimate of the population of this city for 2011?

Solution

This situation can be modelled by a geometric sequence whose first term is 23 140 and whose common ratio is 1.0035. Since we count the population of 1981 among the terms, the number of terms is 31.

2011 is equivalent to the 31st term in this sequence. The estimated population for Baden is, therefore,

$$\text{Population }(2011) = a_{31} = 23\,140(1.0035)^{30} = 25\,697$$

Note: In Chapter 4, more realistic population growth models will be explored and more efficient methods will be developed, as well as the ability to calculate interest that is continuously compounded.

Exercise 3.3

• **Hint:** Definition: In a finite geometric sequence $a_1, a_2, a_3, \ldots, a_k$, the terms $a_2, a_3, \ldots a_{k-1}$ are called **geometric means** between a_1 and a_k.

1 Insert four geometric means between 3 and 96.

2 Determine whether the sequence in each question is arithmetic, geometric or neither. Find the common difference for the arithmetic ones and the common ratio for the geometric ones. Find the common difference or ratio, and the 10th term for each arithmetic or geometric one as appropriate.

a) $a_n = 3n - 3$ b) $b_n = 2^{n+2}$

c) $c_n = 2c_{n-1} - 2$, and $c_1 = -1$ d) $u_n = 3u_{n-1}$ and $u_1 = 4$

e) 2, 5, 12.5, 31.25, 78.125 … f) 2, −5, 12.5, −31.25, 78.125 …

g) 2, 2.75, 3.5, 4.25, 5, … h) $18, -12, 8, -\frac{16}{3}, \frac{32}{9}, \ldots$

For each geometric sequence in questions 3–8, find

a) the 8th term b) an explicit formula for the *nth* term

c) a recursive formula for the *nth* term.

3 $-2, 3, -\frac{9}{2}, \frac{27}{4}, \ldots$ **4** $35, 25, \frac{125}{7}, \frac{625}{49}, \ldots$ **5** $-6, -3, -\frac{3}{2}, -\frac{3}{4}, \ldots$

6 9.5, 19, 38, 76, … **7** 100, 95, 90.25, … **8** $2, \frac{3}{4}, \frac{9}{32}, \frac{27}{256}, \ldots$

9 Find three geometric means between 7 and 4375.

10 Find a geometric *mean* between 16 and 81. • **Hint:** This is also called the **mean proportional**.

11 The first term of a geometric sequence is 24 and the fourth term is 3. Find the fifth term and an expression for the *n*th term.

12 The common ratio in a geometric sequence is $\frac{2}{7}$ and the fourth term is $\frac{14}{3}$. Find the third term.

13 Which term of the geometric sequence 6, 18, 54, … is 118 098?

14 The fourth term and the seventh term of a geometric sequence are 18 and $\frac{729}{8}$. Is $\frac{59049}{128}$ a term of this sequence? If so, which term is it?

15 Jim put €1500 into a savings account that pays 4% interest compounded semi-annually. How much will his account hold 10 years later if he does not make any additional investments in this account?

16 At her daughter Jane's birth, Charlotte set aside £500 into a savings account. The interest she earned was 4% compounded quarterly. How much money will Jane have on her 16th birthday?

17 How much money should you invest now if you wish to have an amount of €4000 in your account after 6 years if interest is compounded quarterly at an annual rate of 5%?

18 In 2007, the population of Switzerland (in thousands) was estimated to be 7554. How large would the Swiss population be in 2012 if it grows at a rate of 0.5% annually?

3.4 Series

The word 'series' in common language implies much the same thing as 'sequence'. But in mathematics when we talk of a series, we are referring in particular to sums of terms in a sequence, e.g. for a sequence of values a_n, the corresponding series is the sequence of S_n with

$$S_n = a_1 + a_2 + \dots + a_{n-1} + a_n$$

If the terms are in an arithmetic sequence, we call the sum an **arithmetic series.**

Sigma notation

Most of the series we consider in mathematics are **infinite** series. This name is used to emphasize the fact that the series contain infinitely many terms. Any sum in the series S_k will be called a partial sum and is given by

$$S_k = a_1 + a_2 + \dots + a_{k-1} + a_k$$

For convenience, this partial sum is written using the sigma notation:

$$S_k = \sum_{i=1}^{i=k} a_i = a_1 + a_2 + \dots + a_{k-1} + a_k$$

Sigma notation is a concise and convenient way to represent long sums. Here, the symbol Σ is the Greek capital letter *sigma* that refers to the initial letter of the word 'sum'. So, the expression $\sum_{i=1}^{i=k} a_i$ means the sum of all the terms a_i, where i takes the values from 1 to k. We can also write $\sum_{i=m}^{n} a_i$ to mean the sum of the terms a_i, where i takes the values from m to n. In such a sum, m is called the lower limit and n the upper limit.

Example 13

Write out what is meant by:

a) $\sum_{i=1}^{5} i^4$ b) $\sum_{r=3}^{7} 3^r$ c) $\sum_{j=1}^{n} x_j p(x_j)$

Solution

a) $\sum_{i=1}^{5} i^4 = 1^4 + 2^4 + 3^4 + 4^4 + 5^4$

b) $\sum_{r=3}^{7} 3^r = 3^3 + 3^4 + 3^5 + 3^6 + 3^7$

c) $\sum_{j=1}^{n} x_j p(x_j) = x_1 p(x_1) + x_2 p(x_2) + \dots + x_n p(x_n)$

Example 14

Evaluate $\sum_{n=0}^{5} 2^n$

Solution

$$\sum_{n=0}^{5} 2^n = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 = 63$$

Example 15

Write the sum $\frac{1}{2} - \frac{2}{3} + \frac{3}{4} - \frac{4}{5} + \ldots + \frac{99}{100}$ in sigma notation.

Solution

We notice that each term's numerator and denominator are consecutive integers, so they take on the absolute value of $\frac{k}{k+1}$ or any equivalent form.

We also notice that the signs of the terms alternate and that we have 99 terms. To take care of the sign, we use some power of (-1) that will start with a positive value. If we use $(-1)^k$, the first term will be negative, so we can use $(-1)^{k+1}$ instead. We can, therefore, write the sum as

$$(-1)^{1+1}\frac{1}{2} + (-1)^{2+1}\frac{2}{3} + (-1)^{3+1}\frac{3}{4} + \ldots + (-1)^{99+1}\frac{99}{100} = \sum_{k=1}^{99}(-1)^{k+1}\frac{k}{k+1}$$

Properties of the sigma notation

There are a number of useful results that we can obtain when we use sigma notation.

1 For example, suppose we had a sum of constant terms

$$\sum_{i=1}^{5} 2$$

What does this mean? If we write this out in full, we get

$$\sum_{i=1}^{5} 2 = 2 + 2 + 2 + 2 + 2 = 5 \times 2 = 10.$$

In general, if we sum a constant n times then we can write

$$\sum_{i=1}^{n} k = k + k + \ldots + k = n \times k = nk.$$

2 Suppose we have the sum of a constant times i. What does this give us? For example,

$$\sum_{i=1}^{5} 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + 5 \times 5 = 5 \times (1 + 2 + 3 + 4 + 5) = 75.$$

However, this can also be interpreted as follows

$$\sum_{i=1}^{5} 5i = 5 \times 1 + 5 \times 2 + 5 \times 3 + 5 \times 4 + 5 \times 5 = 5 \times (1 + 2 + 3 + 4 + 5) = 5\sum_{i=1}^{5} i$$

which implies that

$$\sum_{i=1}^{5} 5i = 5\sum_{i=1}^{5} i$$

In general, we can say

$$\sum_{i=1}^{n} ki = k \times 1 + k \times 2 + \ldots + k \times n$$
$$= k \times (1 + 2 + \ldots + n)$$
$$= k\sum_{i=1}^{n} i$$

3 Suppose that we need to consider the summation of two different functions, such as

$$\sum_{k=1}^{n} (k^2 + k^3) = (1^2 + 1^3) + (2^2 + 2^3) + \ldots + n^2 + n^3$$
$$= (1^2 + 2^2 + \ldots + n^2) + (1^3 + 2^3 + \ldots + n^3)$$
$$= \sum_{k=1}^{n} (k^2) + \sum_{k=1}^{n} (k^3)$$

In general,

$$\sum_{k=1}^{n} (f(k)) + g(k)) = \sum_{k=1}^{n} f(k) + \sum_{k=1}^{n} g(k)$$

Arithmetic series

In arithmetic series, we are concerned with adding the terms of arithmetic sequences. It is very helpful to be able to find an easy expression for the partial sums of this series.

Let us start with an example:

Find the partial sum for the first 50 terms of the series

$$3 + 8 + 13 + 18 + \ldots$$

We express S_{50} in two different ways:

$$S_{50} = 3 + 8 + 13 + \ldots + 248, \text{ and}$$
$$S_{50} = 248 + 243 + 238 + \ldots + 3$$
$$2S_{50} = 251 + 251 + 251 + \ldots + 251$$

There are 50 terms in this sum, and hence

$$2S_{50} = 50 \times 251 \Rightarrow S_{50} = \frac{50}{2}(251) = 6275.$$

This reasoning can be extended to any arithmetic series in order to develop a formula for the nth partial sum S_n.

Let $\{a_n\}$ be an arithmetic sequence with first term a_1 and a common difference d. We can construct the series in two ways: Forward, by adding d to a_1 repeatedly, and backwards by subtracting d from a_n repeatedly. We get the following two expressions for the sum:

$$S_n = a_1 + a_2 + a_3 + \ldots + a_n = a_1 + (a_1 + d) + (a_1 + 2d) + \ldots + (a_1 + (n-1)d)$$

and

$$S_n = a_n + a_{n-1} + a_{n-2} + \ldots + a_1 = a_n + (a_n - d) + (a_n - 2d) + \ldots + (a_n - (n-1)d)$$

By adding, term by term vertically, we get

$$
\begin{array}{rcl}
S_n &=& a_1 + (a_1 + d) + (a_1 + 2d) + \dots + (a_1 + (n-1)d) \\
S_n &=& a_n + (a_n - d) + (a_n - 2d) + \dots + (a_n - (n-1)d) \\
\hline
2S_n &=& (a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \dots + (a_1 + a_n)
\end{array}
$$

Since we have n terms, we can reduce the expression above to

$2S_n = n(a_1 + a_n)$, which can be reduced to

$S_n = \frac{n}{2}(a_1 + a_n)$, which in turn can be changed to give an interesting perspective of the sum,

i.e. $S_n = n\left(\frac{a_1 + a_n}{2}\right)$ is n times the average of the first and last terms!

If we substitute $a_1 + (n-1)d$ for a_n then we arrive at an alternative formula for the sum:

$$S_n = \frac{n}{2}(a_1 + a_1 + (n-1)d) = \frac{n}{2}(2a_1 + (n-1)d)$$

Example 16

Find the partial sum for the first 50 terms of the series

$3 + 8 + 13 + 18 + \dots$

Solution

Using the second formula for the sum, we get

$$S_{50} = \frac{50}{2}(2 \times 3 + (50-1)5) = 25 \times 251 = 6275.$$

Using the first formula requires that we know the nth term. So, $a_{50} = 3 + 49 \times 5 = 248$, which now can be used:

$$S_{50} = 25(3 + 248) = 6275.$$

Geometric series

As is the case with arithmetic series, it is often desirable to find a general expression for the nth partial sum of a geometric series.

Let us start with an example:

Find the partial sum for the first 20 terms of the series

$3 + 6 + 12 + 24 + \dots$

We express S_{20} in two different ways and subtract them:

$$
\begin{array}{rcl}
S_{20} &=& 3 + 6 + 12 + \dots + 1\,572\,864 \\
2S_{20} &=& 6 + 12 + \dots + 1\,572\,864 + 3\,145\,728 \\
\hline
-S_{20} &=& 3 - 3\,145\,728
\end{array}
$$

$\Rightarrow S_{20} = 3\,145\,725$

This reasoning can be extended to any geometric series in order to develop a formula for the nth partial sum S_n.

Let $\{a_n\}$ be a geometric sequence with first term a_1 and a common ratio $r \neq 1$. We can construct the series in two ways as before and using the definition of the geometric sequence, i.e. $a_n = a_{n-1} \times r$, then

$$S_n = a_1 + \mathbf{a_2} + \mathbf{a_3} + \ldots + \mathbf{a_{n-1}} + \mathbf{a_n}, \text{ and}$$

$$rS_n = ra_1 + ra_2 + ra_3 + \ldots + ra_{n-1} + ra_n$$

$$= \mathbf{a_2} + \mathbf{a_3} + \ldots + \mathbf{a_{n-1}} + \mathbf{a_n} + ra_n$$

Now, we subtract the first and last expressions to get

$$S_n - rS_n = a_1 - ra_n \Rightarrow S_n(1 - r) = a_1 - ra_n \Rightarrow \mathbf{S_n = \frac{a_1 - ra_n}{1 - r}}; r \neq 1.$$

This expression, however, requires that r, a_1, as well as a_n be known in order to find the sum. However, using the nth term expression developed earlier, we can simplify this sum formula to

$$Sn = \frac{a_1 - r\mathbf{a_n}}{1 - r} = \frac{a_1 - r\mathbf{a_1 r^{n-1}}}{1 - r} = \frac{a_1(1 - r^n)}{1 - r}; r \neq 1.$$

Example 17

Find the partial sum for the first 20 terms of the series $3 + 6 + 12 + 24 + \ldots$ in the opening example for this section.

Solution

$$S_{20} = \frac{3(1 - 2^{20})}{1 - 2} = \frac{3(1 - 1\,048\,576)}{-1} = 3\,145\,725$$

Infinite geometric series

Consider the series

$$\sum_{k=1}^{n} 2\left(\tfrac{1}{2}\right)^{k-1} = 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

Consider also finding the partial sums for 10, 20 and 100 terms. The sums we are looking for are the partial sums of a geometric series. So,

$$\sum_{k=1}^{10} 2\left(\tfrac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{10}}{1 - \frac{1}{2}} \approx 3.996$$

$$\sum_{k=1}^{20} 2\left(\tfrac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{20}}{1 - \frac{1}{2}} \approx 3.999\,996$$

$$\sum_{k=1}^{100} 2\left(\tfrac{1}{2}\right)^{k-1} = 2 \times \frac{1 - \left(\frac{1}{2}\right)^{100}}{1 - \frac{1}{2}} \approx 4$$

As the number of terms increases, the partial sum appears to be approaching the number 4. This is no coincidence. In the language of limits,

$$\lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\tfrac{1}{2}\right)^{k-1} = \lim_{n\to\infty} 2 \times \frac{1 - \left(\frac{1}{2}\right)^{k}}{1 - \frac{1}{2}} = 2 \times \frac{1 - 0}{\frac{1}{2}} = 4, \text{since } \lim_{n\to\infty} \left(\tfrac{1}{2}\right)^{n} = 0.$$

This type of problem allows us to extend the usual concept of a 'sum' of a **finite** number of terms to make sense of sums in which an **infinite** number of terms is involved. Such series are called **infinite series.**

One thing to be made clear about infinite series is that they are not true sums! The associative property of addition of real numbers allows us to extend the definition of the sum of two numbers, such as $a + b$, to three or four or n numbers, but not to an infinite number of numbers. For example, you can add any specific number of 5s together and get a real number, but if you add an *infinite* number of 5s together, you cannot get a real number! The remarkable thing about infinite series is that, in some cases, such as the example above, the sequence of partial sums (which are true sums) approach a finite limit L. The limit in our example is 4. This we write as

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} (a_1 + a_2 + \ldots + a_n) = L.$$

We say that the series **converges** to L, and it is convenient to define L as the **sum of the infinite series**. We use the notation

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = L.$$

We can, therefore, write the limit above as

$$\sum_{k=1}^{\infty} 2\left(\tfrac{1}{2}\right)^{k-1} = \lim_{n\to\infty} \sum_{k=1}^{n} 2\left(\tfrac{1}{2}\right)^{k-1} = 4.$$

If the series does not have a limit, it **diverges** and does not have a sum.

We are now ready to develop a general rule for **infinite geometric series**. As you know, the sum of the geometric series is given by

$$S_n = \frac{a_1 - ra_n}{1 - r} = \frac{a_1 - ra_1r^{n-1}}{1 - r} = \frac{a_1(1 - r^n)}{1 - r};\ r \neq 1.$$

If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$ and

$$S_n = S = \lim_{n\to\infty} \frac{a_1(1 - r^n)}{1 - r} = \frac{a_1}{1 - r}.$$

We will call this **the sum of the infinite geometric series.** In all other cases the series diverges. The proof is left as an exercise.

$$\sum_{k=1}^{\infty} 2\left(\tfrac{1}{2}\right)^{k-1} = \frac{2}{1 - \frac{1}{2}} = 4, \text{ as already shown.}$$

Example 18

A rational number is a number that can be expressed as a quotient of two integers. Show that $0.\overline{6} = 0.666\ldots$ is a rational number.

Solution

$$0.\overline{6} = 0.666\ldots = 0.6 + 0.06 + 0.006 + 0.0006 + \ldots$$
$$= \tfrac{6}{10} + \tfrac{6}{10}\cdot\tfrac{1}{10} + \tfrac{6}{10}\cdot\left(\tfrac{1}{10}\right)^2 + \tfrac{6}{10}\cdot\left(\tfrac{1}{10}\right)^3 + \ldots$$

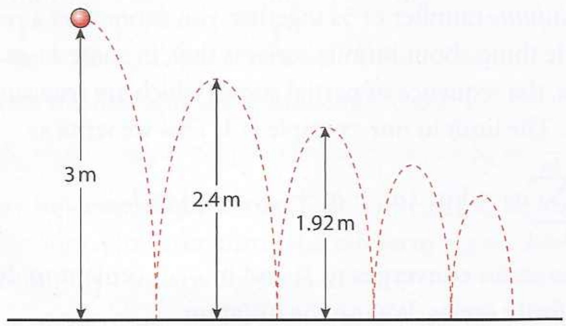
This is an infinite geometric series with $a_1 = \frac{6}{10}$ and $r = \frac{1}{10}$; therefore,

$$0.\overline{6} = \frac{\frac{6}{10}}{1 - \frac{1}{10}} = \frac{6}{10}\cdot\frac{10}{9} = \frac{2}{3}$$

Example 19

If a ball has elasticity such that it bounces up 80% of its previous height, find the total vertical distances travelled down and up by this ball when it is dropped from an altitude of 3 metres. Ignore friction and air resistance.

Solution



After the ball is dropped the initial 3 m, it bounces up and down a distance of 2.4 m. Each bounce after the first bounce, the ball travels 0.8 times the previous height twice – once upwards and once downwards. So, the total vertical distance is given by

$$h = 3 + 2(2.4 + (2.4 \times 0.8) + (2.4 \times 0.8^2) + \ldots) = 3 + 2 \times l$$

The amount in parenthesis is an infinite geometric series with $a_1 = 2.4$ and $r = 0.8$. The value of that quantity is

$$l = \frac{2.4}{1 - 0.8} = 12.$$

Hence, the total distance required is

$$h = 3 + 2(12) = 27 \text{ m}.$$

Applications of series to compound interest calculations (optional)

Annuities

An **annuity** is a sequence of equal periodic payments. If you are saving money by depositing the same amount at the end of each compounding period, the annuity is called **ordinary annuity**. Using geometric series you can calculate the **future value** (FV) of this annuity, which is the amount of money you have after making the last payment.

You invest €1000 at the end of each year for 10 years at a fixed annual interest rate of 6%. See table below.

Table 3.3 Calculating the future value.

Year	Amount invested	Future value
10	1000	1000
9	1000	$1000(1 + 0.06)$
8	1000	$1000(1 + 0.06)^2$
⋮		
1	1000	$1000(1 + 0.06)^9$

The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = 1000 + 1000(1 + 0.06) + 1000(1 + 0.06)^2 + \ldots + 1000(1 + 0.06)^9$$

This sum is a partial sum of a geometric series with $n = 10$ and $r = 1 + 0.06$. Hence,

$$FV = \frac{1000(1 - (1 + 0.06)^{10})}{1 - (1 + 0.06)} = \frac{1000(1-(1 + 0.06)^{10})}{-0.06} = 13\,180.79.$$

```
Plot1 Plot2 Plot3
nMin=1
\U(n)=U(n-1)*(1+
0.06)
 U(nMin)=1000
\V(n)=
 V(nMin)=
\W(n)=
```

```
sum(seq(u(n),n,1,
10)
         13180.79494
```

This result can also be produced with a GDC, as shown.

We can generalize the previous formula in the same manner. Let the periodic payment be R and the periodic interest rate be i, i.e. $i = \frac{r}{n}$. Let the number of periodic payments be m.

Period	Amount invested	Future value
m	R	R
$m - 1$	R	$R(1 + i)$
$m - 2$	R	$R(1 + i)^2$
⋮		
1	R	$R(1 + i)^{m-1}$

Table 3.4 Calculating the future value – formula.

The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = R + R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{m-1}$$

This sum is a partial sum of a geometric series with m terms and $r = 1 + i$. Hence,

$$FV = \frac{R(1 - (1 + i)^m)}{1 - (1 + i)} = \frac{R(1 - (1 + i)^m)}{-i} = R\left(\frac{(1 + i)^m - 1}{i}\right)$$

Note: If the payment is made at the beginning of the period rather than at the end, the annuity is called **annuity due** and the future value after m periods will be slightly different. The table for this situation is given below.

Period	Amount invested	Future value
m	R	$R(1 + i)$
$m - 1$	R	$R(1 + i)^2$
$m - 2$	R	$R(1 + i)^3$
⋮		
1	R	$R(1 + i)^m$

Table 3.5 Calculating the future value (annuity due).

The future value of this investment is the sum of all the entries in the last column, so it is

$$FV = R(1+i) + R(1+i)^2 + \dots + R(1+i)^{m-1} + R(1+i)^m$$

This sum is a partial sum of a geometric series with m terms and $r = 1 + i$. Hence,

$$FV = \frac{R(1+i(1-(1+i)^m)}{1-(1+i)} = \frac{R(1+i-(1+i)^{m+1})}{-i} = R\left(\frac{(1+i)^{m+1}-1}{i} - 1\right)$$

If the previous investment is made at the beginning of the year rather than at the end, then in 10 years we have

$$FV = R\left(\frac{(1+i)^{m+1}-1}{i} - 1\right) = 1000\left(\frac{(1+0.06)^{10+1}-1}{0.006} - 1\right) = 13\,971.64.$$

Exercise 3.4

1 Find the sum of the arithmetic series $11 + 17 + \dots + 365$.

2 Find the sum:

$$2 - 3 + \frac{9}{2} - \frac{27}{4} + \dots - \frac{177\,147}{1024}$$

3 Evaluate $\sum_{k=0}^{13}(2 - 0.3k)$.

4 Evaluate $2 - \frac{4}{5} + \frac{8}{25} - \frac{16}{125} + \dots$

5 Evaluate $\frac{1}{3} + \frac{\sqrt{3}}{12} + \frac{1}{16} + \frac{\sqrt{3}}{64} + \frac{3}{256} + \dots$

6 Express each repeating decimal as a fraction:

a) $0.\overline{52}$ b) $0.\overline{453}$ c) $3.01\overline{37}$

7 At the beginning of every month, Maggie invests £150 in an account that pays 6% annual rate. How much money will there be in the account after six years?

3.5 The binomial theorem

In this section, you will learn about a sequence of numbers called Pascal's triangle and work with one of its applications, the binomial theorem.

A binomial is a polynomial with two terms. For example, $x + y$ is a binomial. In principle, it is easy to raise $x + y$ to any power; but raising it to high powers would be tedious. In this book, we will find a formula that gives the expansion of $(x + y)^n$ for any positive integer n, but we will leave the proof for higher level courses.

Let us look at some special cases of the expansion of $(x + y)^n$:

$(x+y)^0 = 1$

$(x+y)^1 = x + y$

$(x+y)^2 = x^2 + 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$
$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

There are several things that you will have noticed after looking at the expansion:

- There are $n + 1$ terms in the expansion of $(x + y)^n$.
- The degree of each term is n.
- The powers on x begin with n and decrease to 0.
- The powers on y begin with 0 and increase to n.
- The coefficients are symmetric.

For instance, notice how the exponents of x and y behave in the expansion of $(x + y)^5$.

The exponents of x decrease:

$(x + y)^5 = x^{\boxed{5}} + 5x^{\boxed{4}}y + 10x^{\boxed{3}}y^2 + 10x^{\boxed{2}}y^3 + 5x^{\boxed{1}}y^4 + x^{\boxed{0}}y^5$

The exponents of y increase:

$(x + y)^5 = x^5y^{\boxed{0}} + 5x^4y^{\boxed{1}} + 10x^3y^{\boxed{2}} + 10x^2y^{\boxed{3}} + 5xy^{\boxed{4}} + y^{\boxed{5}}$

Using this pattern, we can now proceed to expand any binomial raised to power n: $(x + y)^n$. For example, leaving a blank for the missing coefficients, the expansion for $(x + y)^7$ can be written as

$(x+y)^7$
$= \square x^7 + \square x^6y + \square x^5y^2 + \square x^4y^3 + \square x^3y^4 + \square x^2y^5 + \square xy^6 + \square y^7$

To finish the expansion we need to determine these coefficients. In order to see the pattern, let us look at the coefficients of the expansion we started the section with.

	column 0	column 1	column 2	column 3	column 4	column 5	column 6	
$(x + y)^0$	1							row 0
$(x + y)^1$	1	1						row 1
$(x + y)^2$	1	2	1					row 2
$(x + y)^3$	1	3	3	1				row 3
$(x + y)^4$	1	4	6	4	1			row 4
$(x + y)^5$	1	5	10	10	5	1		row 5
$(x + y)^6$	1	6	15	20	15	6	1	row 6

A triangle like the one above is known as Pascal's triangle. Notice how the first and **second** terms in row **3** give you the **second** term in row **4**; the third and **fourth** terms in row **3** give you the **fourth** term of row **4**; the second and **third** terms in row **5** give you the **third** term in row **6**; and the fifth and **sixth** terms in row **5** give you the **sixth** term in row **6**, and so on. So now we can state the key property of Pascal's triangle.

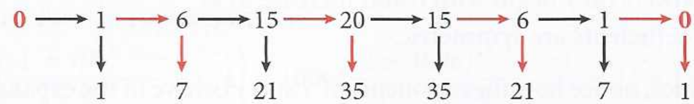
Pascal's triangle

Every entry in a row is the sum of the term directly above it and the entry diagonally above and to the left of it. When there is no entry, the value is considered zero.

Pascal's triangle was known to Persian and Chinese mathematicians in the 13th century.

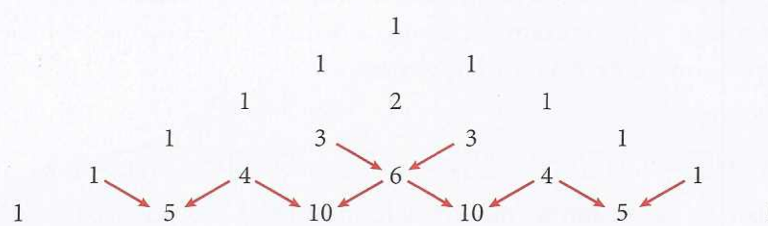
Take the last entry in row 5, for example; there is no entry directly above it, so its value is $0 + 1 = 1$.

From this property it is easy to find all the terms in any row of Pascal's triangle from the row above it. So, for the expansion of $(x + y)^7$, the terms are found from row 6 as follows:



So, $(x+y)^7 = x^7 + \boxed{7}x^6y + \boxed{21}x^5y^2 + \boxed{35}x^4y^3 + \boxed{35}x^3y^4 + \boxed{21}x^2y^5 + \boxed{7}xy^6 + y^7$.

Note: Several sources use a slightly different arrangement for Pascal's triangle. The common usage considers the triangle as isosceles and uses the principle that every two entries add up to give the entry diagonally below them, as shown in the following diagram.



Example 20

Use Pascal's triangle to expand $(2k - 3)^5$.

Solution

We can find the expansion above by replacing x by $2k$ and y by -3 in the binomial expansion of $(x + y)^5$.

Using the fifth row of Pascal's triangle for the coefficients will give us the following:

$$\mathbf{1}(2k)^5 + \mathbf{5}(2k)^4(-3) + \mathbf{10}(2k)^3(-3)^2 + \mathbf{10}(2k)^2(-3)^3 + \mathbf{5}(2k)(-3)^4 + \mathbf{1}(-3)^5 = 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243.$$

Pascal's triangle is an easy and useful tool in finding the coefficients of the binomial expansion for relatively small values of n. It is not very efficient doing that for large values of n. Imagine you want to evaluate $(x + y)^{20}$. Using Pascal's triangle, you will need the terms in the 19th row and the 18th row and so on. This makes the process tedious and not practical.

Luckily, we have a formula that can find the coefficients of any Pascal's triangle row. This formula is the binomial formula, whose proof is beyond the scope of this book. Every entry in Pascal's triangle is denoted by $\binom{n}{r}$, which is also known as the binomial coefficient.

The proof that Pascal's entry and the binomial coefficient are the same can be found by visiting www.heinemann.co.uk/hotlinks and entering the express code 4235P, then clicking on weblink 1.

In $\binom{n}{r}$, n is the row number and r is the column number. To understand the binomial coefficient, we need to understand what the factorial notation means.

> **Factorial notation**
>
> The product of the first n positive integers is denoted by $n!$ and is called ***n* factorial**:
> $$n! = 1 \times 2 \times 3 \times 4 \ldots (n-2) \times (n-1) \times n$$
> We also define $0! = 1$.

This definition of the factorial makes many formulae involving the multiplication of consecutive positive integers shorter and easier to write. That includes the binomial coefficient.

> **The binomial coefficient**
>
> With n and r as non-negative integers such that $n \geqslant r$, the binomial coefficient $\binom{n}{r}$ is defined by
> $$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Note: The GDC uses ${}_nC_r$ to represent $\binom{n}{r}$.

Example 21

Find the value of a) $\binom{7}{3}$ b) $\binom{7}{4}$ c) $\binom{7}{0}$ d) $\binom{7}{7}$

Solution

a) $\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!4!} = \frac{\cancel{1\cdot 2\cdot 3\cdot 4}\cdot 5\cdot 6\cdot 7}{(1\cdot 2\cdot 3)(\cancel{1\cdot 2\cdot 3\cdot 4})} = \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3} = 35$

b) $\binom{7}{4} = \frac{7!}{3!(7-4)!} = \frac{7!}{4!3!} = \frac{\cancel{1\cdot 2\cdot 3\cdot 4}\cdot 5\cdot 6\cdot 7}{(\cancel{1\cdot 2\cdot 3\cdot 4})(1\cdot 2\cdot 3)} = \frac{5\cdot 6\cdot 7}{1\cdot 2\cdot 3} = 35$

c) $\binom{7}{0} = \frac{7!}{0!(7-0)!} = \frac{\cancel{7!}}{0!\cancel{7!}} = \frac{1}{1} = 1$

d) $\binom{7}{7} = \frac{7!}{7!(7-7)!} = \frac{\cancel{7!}}{\cancel{7!}0!} = \frac{1}{1} = 1$

- **Hint:** Your calculator can do the tedious work of evaluating the binomial coefficient. If you have a TI, the binomial coefficient appears as ${}_nC_r$, which is another notation frequently used in mathematical literature.

```
7 nCr 3
                    35
7 nCr 4
                    35
7 nCr 0
                     1
█
```

Although the binomial coefficient $\binom{n}{r}$ appears as a fraction, all its results where n and r are non-negative integers are positive integers. Also, notice the **symmetry** of the coefficient in the previous examples. This is a property that you are asked to prove in the exercises:

$$\binom{n}{r} = \binom{n}{n-r}$$

Example 22

Calculate the following:

$\binom{6}{0}, \binom{6}{1}, \binom{6}{2}, \binom{6}{3}, \binom{6}{4}, \binom{6}{5}, \binom{6}{6}$

Solution

$\binom{6}{0} = 1$, $\binom{6}{1} = 6$, $\binom{6}{2} = 15$, $\binom{6}{3} = 20$, $\binom{6}{4} = 15$, $\binom{6}{5} = 6$, $\binom{6}{6} = 1$

The values we calculated above are precisely the entries in the sixth row of Pascal's triangle.

We can write Pascal's triangle in the following manner:

$$\begin{array}{lllllll} \binom{0}{0} \\ \binom{1}{0} & \binom{1}{1} \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\ \cdots & \cdots & \cdots & \cdots \\ \binom{n}{0} & \binom{n}{1} & \cdots & \cdots & \cdots & \cdots & \binom{n}{n} \end{array}$$

Example 23

Calculate $\binom{n}{r-1} + \binom{n}{r}$.

• **Hint:** You will be able to provide reasons for the steps after you do the exercises!

Solution

$$\begin{aligned} \binom{n}{r-1} + \binom{n}{r} &= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!} \\ &= \frac{n! \cdot r}{r \cdot (r-1)!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r)! \cdot (n-r+1)} \\ &= \frac{n! \cdot r}{r!(n-r+1)!} + \frac{n! \cdot (n-r+1)}{r!(n-r+1)!} \\ &= \frac{n! \cdot r + n! \cdot (n-r+1)}{r!(n-r+1)!} = \frac{n!(r+n-r+1)}{r!(n-r+1)!} \\ &= \frac{n!(n+1)}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n+1-r)!} = \binom{n+1}{r} \end{aligned}$$

If we read the result above carefully, it says that the sum of the terms in the nth row $(r-1)$th and rth columns is equal to the entry in the $(n+1)$th row and rth column. That is, the two entries on the left are adjacent entries in the nth row of Pascal's triangle and the entry on the right is the entry in the $(n+1)$th row directly below the rightmost entry. This is precisely the principle behind Pascal's triangle!

Using the binomial theorem

We are now prepared to state the binomial theorem.

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \binom{n}{3}x^{n-3}y^3 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

In a compact form, we can use sigma notation to express the theorem as follows:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

Example 24

Use the binomial theorem to expand $(x + y)^7$.

Solution

$$(x+y)^7 = \binom{7}{0}x^7 + \binom{7}{1}x^{7-1}y + \binom{7}{2}x^{7-2}y^2 + \binom{7}{3}x^{7-3}y^3 + \binom{7}{4}x^{7-4}y^4 + \binom{7}{5}x^{7-5}y^5 + \binom{7}{6}xy^6 + \binom{7}{7}y^7$$

$$= x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$$

Example 25

Find the expansion for $(2k - 3)^5$.

Solution

$$(2k-3)^5 = \binom{5}{0}(2k)^5 + \binom{5}{1}(2k)^4(-3) + \binom{5}{2}(2k)^3(-3)^2 + \binom{5}{3}(2k)^2(-3)^3 + \binom{5}{4}(2k)(-3)^4 + \binom{5}{5}(-3)^5$$

$$= 32k^5 - 240k^4 + 720k^3 - 1080k^2 + 810k - 243$$

Example 26

Find the term containing a^3 in the expansion $(2a - 3b)^9$.

Solution

To find the term, we do not need to expand the whole expression.

Since $(x + y)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}y^i$, the term containing a^3 is the term where $n - i = 3$, i.e. when $i = 6$. So, the required term is

$$\binom{9}{6}(2a)^{9-6}(-3b)^6 = 84 \cdot 8a^3 \cdot 729b^6 = 489\,888a^3b^6.$$

Example 27

Find the term independent of x in $\left(2x^2 - \frac{3}{x}\right)^9$.

Solution

To get such a term, we need the power of the first term to be equal to the power of the second term, i.e. $2k = 9 - k \Rightarrow k = 3$, so the term is

$$\binom{9}{6}(2x^2)^3\left(-\frac{3}{x}\right)^6 = 84 \cdot 8x^6 \cdot \frac{729}{x^6} = 489\,888.$$

Example 28

Find the coefficient of b^6 in the expansion of $\left(2b^2 - \frac{1}{b}\right)^{12}$.

Solution

The general term is

$$\binom{12}{i}(2b^2)^{12-i}\left(-\frac{1}{b}\right)^i = \binom{12}{i}(2)^{12-i}(b^2)^{12-i}\left(-\frac{1}{b}\right)^i$$

$$= \binom{12}{i}(2)^{12-i}b^{24-2i}b^{-i}(-1)^i = \binom{12}{i}(2)^{12-i}b^{24-3i}(-1)^i$$

$24 - 3i = 6 \Rightarrow i = 6$. So, the coefficient in question is $\binom{12}{6}(2)^6(-1)^6 = 59\,136$.

Exercise 3.5

1 Use Pascal's triangle to expand each binomial.

a) $(x + 2y)^5$ b) $(a - b)^4$ c) $(x - 3)^6$

d) $(2 - x^3)^4$ e) $(x - 3b)^7$ f) $\left(2n + \frac{1}{n^2}\right)^6$

g) $\left(\frac{3}{x} - 2\sqrt{x}\right)^4$

2 Evaluate each expression.

a) $\binom{8}{3}$ b) $\binom{18}{5} - \binom{18}{13}$ c) $\binom{7}{4}\binom{7}{3}$

d) $\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$

e) $\binom{6}{0} - \binom{6}{1} + \binom{6}{2} - \binom{6}{3} + \binom{6}{4} - \binom{6}{5} + \binom{6}{6}$

3 Use the binomial theorem to expand each of the following.

a) $(x + 2y)^7$ b) $(a - b)^6$ c) $(x - 3)^5$

d) $(2 - x^3)^6$ e) $(x - 3b)^7$ f) $\left(2n + \frac{1}{n^2}\right)^6$

g) $\left(\frac{3}{x} - 2\sqrt{x}\right)^4$ h) $(1 + \sqrt{5})^4 + (1 - \sqrt{5})^4$

i) $(\sqrt{3} + 1)^8 - (\sqrt{3} - 1)^8$ j) $(1 + i)^8$, where $i^2 = -1$

k) $(\sqrt{2} - i)^6$, where $i^2 = -1$

4 Consider the expression $\left(x - \frac{2}{x}\right)^{45}$.

a) Find the first three terms of this expansion.
b) Find the constant term if it exists or justify why it does not exist.
c) Find the last three terms of the expansion.
d) Find the term containing x^3 if it exists or justify why it does not exist.

5 Prove that $\binom{n}{k} = \binom{n}{n-k}$ for all $n, k \in \mathbb{N}$ and $n \geq k$.

6 Prove that for any positive integer n,

$\binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n - 1$ • **Hint:** $2^n = (1 + 1)^n$

7 Consider all $n, k \in \mathbb{N}$ and $n \geq k$.

a) Verify that $k! = k(k - 1)!$
b) Verify that $(n - k + 1)! = (n - k + 1)(n - k)!$
c) Justify the steps given in the proof of $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$ in the examples.

8 Find the value of the expression:

$$\binom{6}{0}\left(\frac{1}{3}\right)^6 + \binom{6}{1}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \binom{6}{2}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + \ldots + \binom{6}{6}\left(\frac{2}{3}\right)^6$$

9 Find the value of the expression:

$$\binom{8}{0}\left(\frac{2}{5}\right)^8 + \binom{8}{1}\left(\frac{2}{5}\right)^7\left(\frac{3}{5}\right) + \binom{8}{2}\left(\frac{2}{5}\right)^6\left(\frac{3}{5}\right)^2 + \ldots + \binom{8}{8}\left(\frac{3}{5}\right)^8$$

10 Find the value of the expression:

$$\binom{n}{0}\left(\frac{1}{7}\right)^n + \binom{n}{1}\left(\frac{1}{7}\right)^{n-1}\left(\frac{6}{7}\right) + \binom{n}{2}\left(\frac{1}{7}\right)^{n-2}\left(\frac{6}{7}\right)^2 + \ldots + \binom{n}{n}\left(\frac{6}{7}\right)^n$$

Practice questions

Find the first five terms of each infinite sequence defined in questions 1–6.

1 $s(n) = 2n - 3$

2 $g(k) = 2^k - 3$

3 $f(n) = 3 \times 2^{-n}$

4 $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 3; \quad \text{for } n > 1 \end{cases}$

5 $a_n = (-1)^n(2^k) + 3$

6 $\begin{cases} b_1 = 3 \\ b_n = b_{n-1} + 2n; \quad \text{for } n \geqslant 2 \end{cases}$

Determine whether each sequence in questions 7–12 is arithmetic, geometric or neither. Find the common difference for the arithmetic ones and the common ratio for the geometric ones.

7 52, 55, 58, 61, …

8 −1, 3, −9, 27, −81, …

9 0.1, 0.2, 0.4, 0.8, 1.6, 3.2, …

10 3, 6, 12, 18, 21, 27, …

11 6, 14, 20, 28, 34, …

12 2.4, 3.7, 5, 6.3, 7.6, …

For each arithmetic or geometric sequence in questions 13–23, find

a) the 8th term

b) an explicit formula for the *n*th term

c) a recursive formula for the *n*th term.

13 −3, 2, 7, 12, …

14 19, 15, 11, 7, …

15 −8, 3, 14, 25, …

16 10.05, 9.95, 9.85, 9.75, …

17 100, 99, 98, 97, …

18 $2, \frac{1}{2}, -1, -\frac{5}{2}, \ldots$

19 3, 6, 12, 24, …

20 4, 12, 36, 108, …

21 5, −5, 5, −5, …

22 3, −6, 12, −24, …

23 972, −324, 108, −361, …

24 Find five arithmetic means between 15 and −21.

25 Find three arithmetic means between 99 and 100.

26 In an arithmetic sequence, $a_3 = 11$ and $a_{12} = 47$. Find an explicit formula for the nth term of this sequence.

27 In an arithmetic sequence, $a_7 = -48$ and $a_{13} = -10$. Find an explicit formula for the nth term of this sequence.

28 Find four geometric means between 7 and 1701.

29 Find a geometric mean between 9 and 64. • **Hint:** This is also called the **mean proportional.**

30 The first term of a geometric sequence is 24 and the third term is 6. Find the fourth term and an expression for the nth term.

31 The common ratio in a geometric sequence is $\frac{3}{7}$ and the fourth term is $\frac{14}{3}$. Find the third term.

32 Which term of the geometric sequence 7, 21, 63, ... is 137 781?

33 The third term and the sixth term of a geometric sequence are 18 and $\frac{243}{4}$. Is $\frac{19\,683}{64}$ a term of this sequence? If so, which term is it?

34 Tim put €2500 into a savings account that pays 4% interest compounded semi-annually. How much will his account hold 10 years later if he does not make any additional investments in this account?

35 At her son William's birth, Jane set aside £1000 into a savings account. The interest she earned was 6% compounded quarterly. How much money will William have on his 18th birthday?

36 How much money should you invest now if you wish to have an amount of €3000 in your account after six years if interest is compounded quarterly at an annual rate of 6%?

37 Find the sum of the arithmetic series $13 + 19 + \ldots + 367$.

38 Find the sum of

$$2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots - \frac{4096}{177\,147}$$

39 Evaluate $\sum_{k=0}^{11}(3 + 0.2k)$.

40 Evaluate $2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots$

41 Evaluate $\frac{1}{2} + \frac{\sqrt{2}}{2\sqrt{3}} + \frac{1}{3} + \frac{\sqrt{2}}{3\sqrt{3}} + \frac{2}{9} + \ldots$

42 Express each repeating decimal as a fraction:

a) $0.\overline{7}$ **b)** $0.3\overline{45}$ **c)** $3.21\overline{29}$

43 Find the coefficient of x^6 in the expansion of $(2x - 3)^9$.

44 Find the coefficient of x^3b^4 in $(ax + b)^7$.

45 Find the constant term of $\left(\frac{2}{z^2} - z\right)^{15}$.

46 Expand $(3n - 2m)^5$.

47 Find the coefficient of r^{10} in $(4 + 3r^2)^9$.

48 In an arithmetic sequence, the first term is 4, the fourth term is 19 and the nth term is 99. Find the common difference and the number of terms n.

49 Two students, Nick and Charlotte, decide to start preparing for their IB exams 15 weeks ahead of the exams. Nick starts by studying for 12 hours in the first week and plans to increase the amount by 2 hours per week. Charlotte starts with 12 hours in the first week and decides to increase her time by 10% every week.

a) How many hours did each student study in week 5?

b) How many hours in total does each student study for the 15 weeks?

c) In which week will Charlotte exceed 40 hours per week?

d) In which week does Charlotte catch up with Nick in the number of hours spent on studying per week?

50 Two diet schemes are available for relatively overweight people to lose weight. Plan A promises the patient an initial weight loss of 1000 g the first month, with a steady loss of an additional 80 g every month after the first. So, the second month the patient will lose 1080 g and so on for a maximum duration of 12 months.

Plan B starts with a weight loss of 1000 g the first month and an increase in weight loss by 6% more every following month.

a) Write down the amount of grams lost under Plan B in the second and third months.

b) Find the weight lost in the 12th month for each plan.

c) Find the total weight loss during a 12-month period under

(i) Plan A **(ii)** Plan B.

51 Planning on buying your first car in 10 years, you start a savings plan where you invest €500 at the beginning of the year for 10 years. Your investment scheme offers a fixed rate of 6% per year compounded annually.

Calculate, giving your answers to the nearest euro (€),

(a) how much the first €500 is worth at the end of 10 years

(b) the total value your investment will give you at the end of the 10 years.

52 The first three terms of an arithmetic sequence are 6, 9.5, 13.

a) What is the 40th term of the sequence?

b) What is the sum of the first 103 terms of the sequence?

53 A marathon runner plans her training programme for a 20 km race. On the first day she plans to run 2 km, and then she wants to increase her distance by 500 m on each subsequent training day.

a) On which day of her training does she first run a distance of 20 km?

b) By the time she manages to run the 20 km distance, what is the total distance she would have run for the whole training programme?

54 In the nation of Telefonica, cellular phones were first introduced in the year 2000. During the first year, the number of people who bought a cellular phone was 1600. In 2001, the number of new participants was 2400, and in 2002 the new participants numbered 3600.

a) You notice that the trend is a geometric sequence; find the common ratio.

Assuming that the trend continues,

b) how many participants will join in 2012?

c) in what year would the number of new participants first exceed 50 000?

Between 2000 and 2002, the total number of participants reaches 7600.

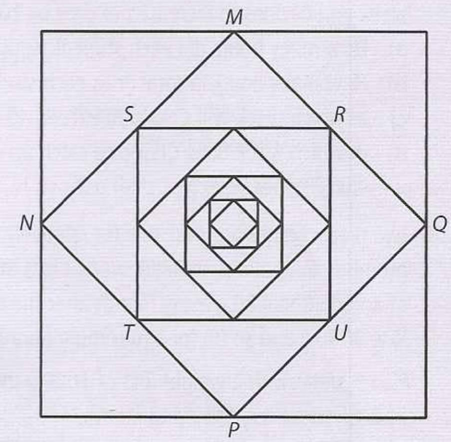
d) What is the total number of participants between 2000 and 2012?

During this period, the total adult population of Telefonica remains at approximately 800 000.

e) Use this information to suggest a reason why this trend in growth would not continue.

55 In an arithmetic sequence, the fist term is 25, the fourth term is 13 and the nth term is $-11\,995$. Find the common difference d and the number of terms n.

56 The midpoints M, N, P, Q of the sides of a square of side 1 cm are joined to form a new square.



a) Show that the side of the second square $MNPQ$ is $\frac{\sqrt{2}}{2}$.

b) Find the area of square $MNPQ$.

A new third square $RSTU$ is constructed in the same manner.

c) **(i)** Find the area of the third square just constructed.

(ii) Show that the areas of the squares are in a geometric sequence and find its common ratio.

The procedure continues indefinitely.

d) **(i)** Find the area of the tenth square.

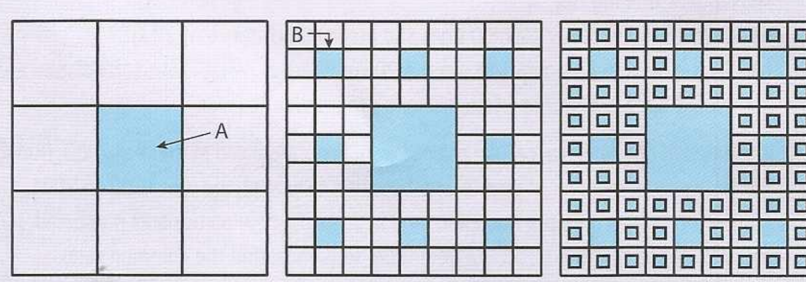
(ii) Find the sum of the areas of all the squares.

57 Tim is a dedicated swimmer. He goes swimming once every week. He starts the first week of the year by swimming 200 metres. Each week after that he swims 20 m more than the previous week. He does that all year long (52 weeks).

a) How far does he swim in the final week?

b) How far does he swim altogether?

58 The diagram below shows three iterations of constructing squares in the following manner: A square of side 3 units is given, then it is divided into nine smaller squares as shown and the middle square is shaded. Each of the unshaded squares is in turn divided into nine squares and the process is repeated. The area of the first shaded square is 1 unit.



a) Find the area of each of the squares A and B.

b) Find the area of any small square in the third diagram.

c) Find the area of the shaded regions in the second and third iterations.

d) If the process is continued indefinitely, find the area left unshaded.

59 The table below shows four series of numbers. One series is an arithmetic one, one is a converging geometric series, one is a diverging geometric series and the fourth is neither geometric nor arithmetic.

Series		Type of series
(i)	$2 + 22 + 222 + 2222 + \dots$	
(ii)	$2 + \frac{4}{3} + \frac{8}{9} + \frac{16}{27} + \dots$	
(iii)	$0.8 + 0.78 + 0.76 + 0.74 + \dots$	
(iv)	$2 + \frac{8}{3} + \frac{32}{9} + \frac{128}{27} + \dots$	

a) Complete the table by stating the type of each series.

b) Find the sum of the infinite geometric series above.

60 Two IT companies offer 'apparently' similar salary schemes for their new appointees. Kell offers a starting salary of €18 000 per year and then an annual increase of €400 every year after the first. YBO offers a starting salary of €17 000 per year and an annual increase of 7% for the rest of the years after the first.

a) **(i)** Write down the salary paid during the second and third years for each company.

(ii) Calculate the total amount that an employee working for 10 years will accumulate in each company.

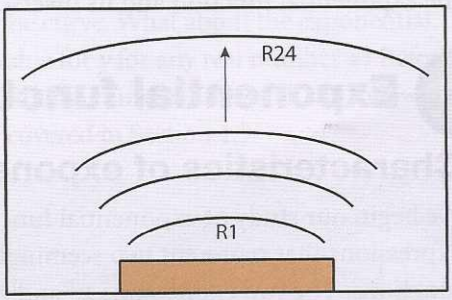
(iii) Calculate the salary paid during the tenth year for each company.

b) Tim works at Kell and Merijayne works at YBO.

(i) When would Merijayne start earning more than Tim?

(ii) What is the minimum number of years that Merijayne requires so that her total earnings exceed Tim's total earnings?

61 A theatre has 24 rows of seats. There are 16 seats in the first row and each successive row increases by 2 seats, 1 on each side.



a) Calculate the number of seats in the 24th row.

b) Calculate the number of seats in the whole theatre.

62 The amount of €7000 is invested at 5.25% annual compound interest.

a) Write down an expression for the value of this investment after *t* full years.

b) Calculate the minimum number of years required for this amount to become €10 000.

c) For the same number of years as in part b), would an investment of the same amount be better if it were at a 5% rate compounded quarterly?

63 With S_n denoting the sum of the first n terms of an arithmetic sequence, we are given that $S_1 = 9$ and $S_2 = 20$.

a) Find the second term.

b) Calculate the common difference of the sequence.

c) Find the fourth term.