4 Exponential and Logarithmic Functions

Assessment statements

1.2 Exponents and logarithms.
Laws of exponents; laws of logarithms. Change of base.

2.7 The function $x \mapsto a^x$, $a > 0$.
The inverse function $x \mapsto \log_a x$, $x > 0$.
Graphs of $y = a^x$ and $y = \log_a x$.
Solutions of $a^x = b$ using logarithms.

2.8 The exponential function $x \mapsto e^x$.
The logarithmic function $x \mapsto \ln x$, $x > 0$.

Introduction

A variety of functions have already been considered in this text (see Figure 2.15 in Section 2.4): polynomial functions (e.g. linear, quadratic and cubic functions), functions with radicals (e.g. square root function), rational functions (e.g. inverse and inverse square functions) and the absolute value functions. This chapter examines two very important and useful functions: the exponential function and its inverse function, the logarithmic function.

4.1 Exponential functions

Characteristics of exponential functions

We begin our study of exponential functions by comparing two algebraic expressions that represent two seemingly similar but very different functions. The two expressions x^2 and 2^x are similar in that they both contain a **base** and an **exponent** (or power). In x^2, the base is the variable x and the exponent is the constant 2. In 2^x, the base is the constant 2 and the exponent is the variable x. However, x^2 and 2^x are examples of two different types of functions.

• **Hint:** Another word for exponent is **index** (plural: **indices**).

The quadratic function $y = x^2$ is in the form 'variable base$^{\text{constant power}}$', where the base is a variable and the exponent is an integer greater than or equal to zero (non-negative integer). Any function in this form is called a **polynomial** (or **power**) **function.**

The function $y = 2^x$ is in the form 'constant base$^{\text{variable power}}$', where the base is a positive real number (not equal to one) and the exponent is a variable. Any function in this form is called an **exponential function.**

To illustrate a fundamental difference between exponential functions and power functions, consider the function values for $y = x^2$ and $y = 2^x$ when

x is an integer from 0 to 10. Both a table and a graph (Figure 4.1) showing these results display clearly how the values for the exponential function eventually increase at a significantly faster rate than the power function.

x	$y = x^2$	$y = 2^x$
0	0	1
1	1	2
2	4	4
3	9	8
4	16	16
5	25	32
6	36	64
7	49	128
8	64	256
9	81	512
10	100	1024

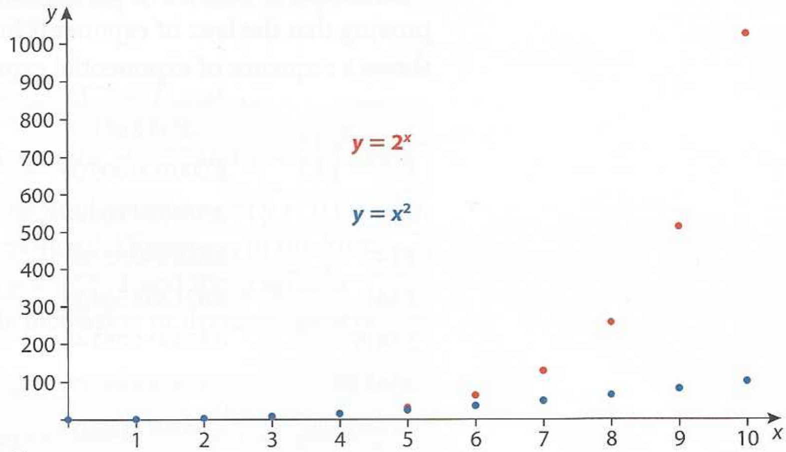

Figure 4.1

Another important point to make is that polynomial, or power, functions can easily be defined (and computed) for any real number. For any power function $y = x^n$, where n is any positive integer, y is found by simply taking x and repeatedly multiplying it n times. Hence, x can be any real number. For example, for the power function $y = x^3$, if $x = \pi$, then $y = \pi^3 \approx 31.006\,276\,68\ldots$. Since a power function like $y = x^3$ is defined for all real numbers, we can graph it as a continuous curve so that every real number is the x-coordinate of some point on the curve. What about the exponential function $y = 2^x$? Can we compute a value for y for any real number x? Before we try, let's first consider x being any rational number and recall the following laws of exponents (indices) that were covered in Section 1.3.

Laws of exponents

For $b > 0$ and $m, n \in \mathbb{Q}$ (rational numbers):

$b^m \cdot b^n = b^{m+n}$ $\quad \dfrac{b^m}{b^n} = b^{m-n}$ $\quad (b^m)^n = b^{mn}$ $\quad b^0 = 1$ $\quad b^{-m} = \dfrac{1}{b^m}$

Also, in Section 1.3, we covered the definition of a rational exponent.

Rational exponent

For $b > 0$ and $m, n \in \mathbb{Z}$ (integers):

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$$

From these established facts, we are able to compute b^x $(b > 0)$ when x is any rational number. For example, $b^{4.7} = b^{\frac{47}{10}}$ represents the 10th root of b raised to the 47th power. Now, we would like to define b^x when x is any real number such as π or $\sqrt{2}$. We know that π has a non-terminating, non-repeating decimal representation that begins $\pi = 3.141\,592\,653\,589\,793\ldots$. Consider the sequence of numbers

$$b^3, b^{3.1}, b^{3.14}, b^{3.141}, b^{3.1415}, b^{3.14159}, \ldots$$

To demonstrate just how quickly $y = 2^x$ increases, consider what would happen if you were able to repeatedly fold a piece of paper in half 50 times. A typical piece of paper is about five thousandths of a centimetre thick. Each time you fold the piece of paper the thickness of the paper doubles, so after 50 folds the thickness of the folded paper is the height of a stack of 2^{50} pieces of paper. The thickness of the paper after being folded 50 times would be $2^{50} \times 0.005$ cm – which is more than 56 million kilometres (nearly 35 million miles)! Compare that with the height of a stack of 50^2 pieces of paper that would be a meagre $12\frac{1}{2}$ cm – only 0.000 125 km.

Every term in this sequence is defined because each has a rational exponent. Although it is beyond the scope of this text, it can be proved that each number in the sequence gets closer and closer to a certain real number – defined as b^{π}. Similarly, we can define other irrational exponents, thus proving that the laws of exponents hold for all real exponents. Figure 4.2 shows a sequence of exponential expressions approaching the value of 2^{π}.

x	2^x (12 s.f.)
3	8.000 000 000 00
3.1	8.574 187 700 29
3.14	8.815 240 927 01
3.141	8.821 353 304 55
3.1415	8.824 411 082 48
3.141 59	8.824 961 595 06
3.141 592	8.824 973 829 06
3.141 5926	8.824 977 499 27
3.141 592 65	8.824 977 805 12

Your GDC will give an approximate value for 2^{π} to at least 10 significant figures, as shown below.

```
2^π
            8.824977827
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Figure 4.2

Graphs of exponential functions

Using this definition of irrational powers, we can now construct a complete graph of any exponential function $f(x) = b^x$ such that b is a number greater than zero and x is any real number.

Example 1

Graph each exponential function by plotting points.

a) $f(x) = 3^x$ b) $g(x) = \left(\frac{1}{3}\right)^x$

Solution

We can easily compute values for each function for integral values of x from -3 to 3. Knowing that exponential functions are defined for all real numbers – not just integers – we can sketch a smooth curve in Figure 4.3, filling in between the ordered pairs shown in the table.

x	$f(x) = 3^x$	$g(x) = \left(\frac{1}{3}\right)^x$
-3	$\frac{1}{27}$	27
-2	$\frac{1}{9}$	9
-1	$\frac{1}{3}$	3
0	1	1
1	3	$\frac{1}{3}$
2	9	$\frac{1}{9}$
3	27	$\frac{1}{27}$

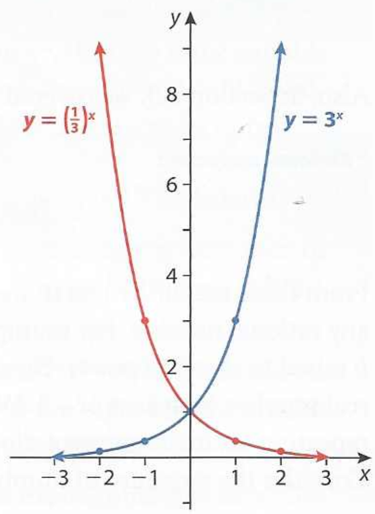


Figure 4.3

Remember that in Section 2.4 we established that the graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ in the y-axis. It is clear from the table and the graph in Figure 4.3 that the graph of function g is a reflection of function f about the y-axis. Let's use some laws of exponents to show that $g(x) = f(-x)$.

$$g(x) = \left(\frac{1}{3}\right)^x = \frac{1^x}{3^x} = \frac{1}{3^x} = 3^{-x} = f(-x)$$

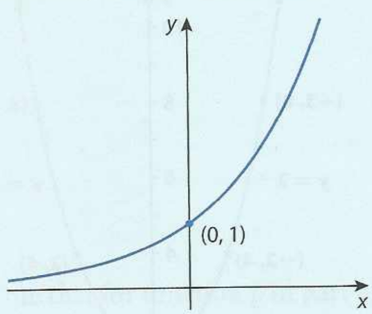
It is useful to point out that both of the graphs, $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$, pass through the point (0, 1) and have a horizontal asymptote of $y = 0$ (x-axis). The same is true for the graph of all exponential functions in the form $y = b^x$ given that $b \neq 1$. If $b = 1$, then $y = 1^x = 1$ and the graph is a horizontal line rather than a constantly increasing or decreasing curve.

The exponential function

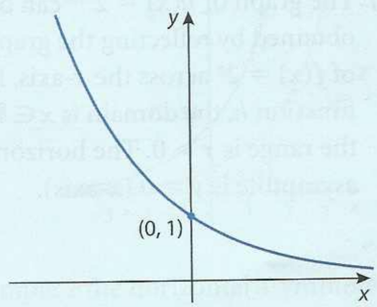
If $b > 0$ and $b \neq 1$, the **exponential function** with base b is the function defined by

$$f(x) = b^x$$

The **domain** of f is the set of real numbers ($x \in \mathbb{R}$) and the **range** of f is the set of positive real numbers ($y > 0$). The graph of f passes through (0, 1), has the x-axis as a **horizontal asymptote**, and, depending on the value of the base of the exponential function b, will either be a continually increasing **exponential growth curve** or a continually decreasing **exponential decay curve**.



$f(x) = b^x$ for $b > 1$
as $x \to \infty$, $f(x) \to \infty$
f is an increasing function
exponential growth curve



$f(x) = b^x$ for $0 < b < 1$
as $x \to \infty$, $f(x) \to 0$
f is a decreasing function
exponential decay curve

The graphs of all exponential functions will display a characteristic growth or decay curve. As we shall see, many natural phenomena exhibit exponential growth or decay. Also, the graphs of exponential functions behave **asymptotically** for either very large positive values of x (decay curve) or very large negative values of x (growth curve). This means that there will exist a horizontal line that the graph will approach, but not intersect, as either $x \to \infty$ or as $x \to -\infty$.

Transformations of exponential functions

Recalling from Section 2.4 how the graphs of functions are translated and reflected, we can efficiently sketch the graph of many exponential functions.

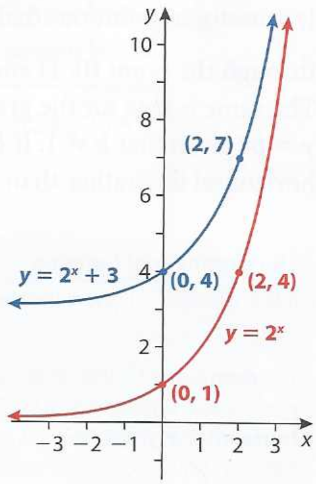
Example 2

Using the graph of $f(x) = 2^x$, sketch the graph of each function. State the domain and range for each function and the equation of its horizontal asymptote.

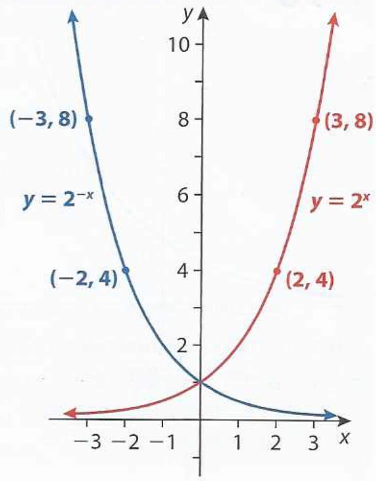
a) $g(x) = 2^x + 3$ b) $h(x) = 2^{-x}$ c) $p(x) = -2^x$
d) $r(x) = 2^{x-4}$ e) $v(x) = 3(2^x)$

Solution

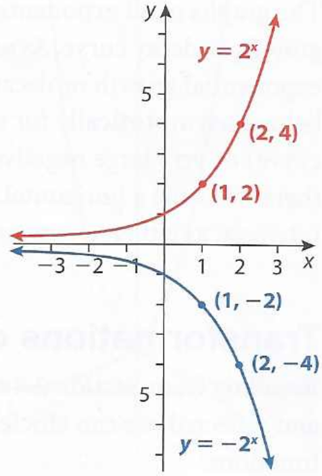
a) The graph of $g(x) = 2^x + 3$ can be obtained by translating the graph of $f(x) = 2^x$ vertically three units up. For function g, the domain is x is any real number ($x \in \mathbb{R}$) and the range is $y > 3$. The horizontal asymptote for g is $y = 3$.



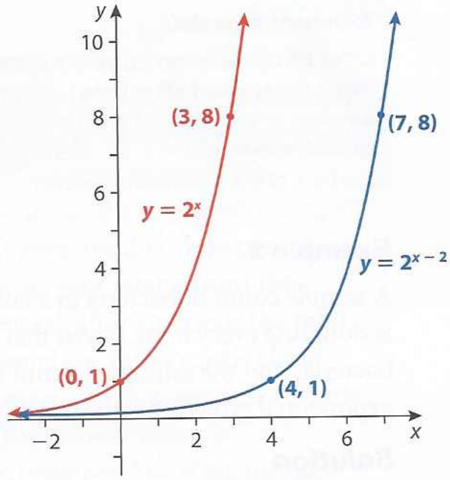
b) The graph of $h(x) = 2^{-x}$ can be obtained by reflecting the graph of $f(x) = 2^x$ across the x-axis. For function h, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



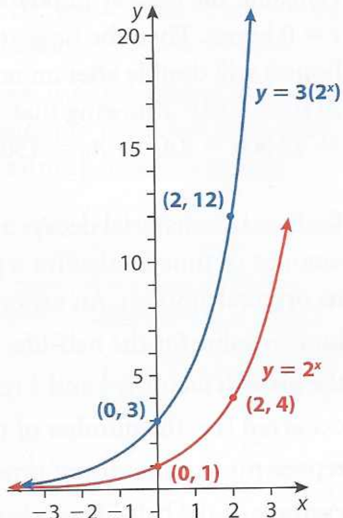
c) The graph of $p(x) = -2^x$ can be obtained by reflecting the graph of $f(x) = 2^x$ across the x-axis. For function p, the domain is $x \in \mathbb{R}$ and the range is $y < 0$. The horizontal asymptote is $y = 0$ (x-axis).



d) The graph of $r(x) = 2^{x-4}$ can be obtained by translating the graph of $f(x) = 2^x$ four units to the right. For function r, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



e) The graph of $v(x) = 3(2^x)$ can be obtained by a vertical stretch of the graph of $f(x) = 2^x$ by scale factor 3. For function v, the domain is $x \in \mathbb{R}$ and the range is $y > 0$. The horizontal asymptote is $y = 0$ (x-axis).



Note that for function p in part c) of Example 2 the horizontal asymptote is an **upper bound** (i.e. no function value is equal to or greater than $y = 0$). Whereas, in parts a), b), d) and e) the horizontal asymptote for each function is a **lower bound** (i.e. no function value is equal to or less than the y-value of the asymptote).

4.2 Exponential growth and decay

Mathematical models of growth and decay

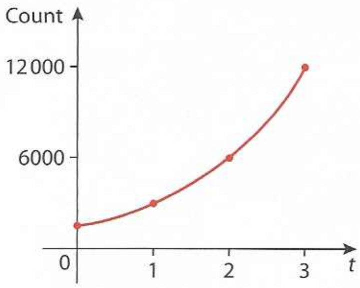
Exponential functions are well suited as a mathematical model for a wide variety of steadily increasing or decreasing phenomena of many kinds, including population growth (or decline), investment of money at compound interest and radioactive decay. Recall from the previous chapter that the formula for finding terms in a geometric sequence (repeated multiplication by common ratio r) is an exponential function. Many instances of growth or decay occur geometrically (repeated multiplication by a growth or decay factor).

Exponential models

Exponential models are equations of the form $A(t) = A_0b^t$, where $A_0 \neq 0$, $b > 0$ and $b \neq 1$. $A(t)$ is the **amount after time t**. $A(0) = A_0b^0 = A_0(1) = A_0$, so A_0 is called the **initial amount** or value (often the value at time $(t) = 0$). If $b > 1$, then $A(t)$ is an **exponential growth model**. If $0 < b < 1$, then $A(t)$ is an **exponential decay model**. The value of b, the base of the exponential function, is often called the **growth or decay factor**.

Example 3

A sample count of bacteria in a culture indicates that the number of bacteria is doubling every hour. Given that the estimated count at 15:00 was 12 000 bacteria, find the estimated count three hours earlier at 12:00 and write an exponential growth function for the number of bacteria at any hour t.



Solution

Consider the time at 12:00 to be the starting, or initial, time and label it $t = 0$ hours. Then the time at 15:00 is $t = 3$. The amount at any time t (in hours) will double after an hour so the growth factor, b, is 2. Therefore, $A(t) = A_0(2)^t$. Knowing that $A(3) = 12\,000$, compute A_0: $12\,000 = A_0(2)^3$ $\Rightarrow 12\,000 = 8A_0 \Rightarrow A_0 = 1500$.

Radioactive material decays at exponential rates. The **half-life** is the amount of time it takes for a given amount of material to decay to half of its original amount. An exponential function that models decay with a known value for the half-life, h, will be of the form $A(t) = A_0\left(\frac{1}{2}\right)^k$, where the growth factor is $\frac{1}{2}$ and k represents the number of half-lives that have occurred (i.e. the number of times that A_0 is multiplied by $\frac{1}{2}$). If t represents the amount of time, the number of half-lives will be $\frac{t}{h}$. For example, if the half-life of a certain material is 25 days and the amount of time that has passed since measuring the amount A_0 is 75 days, then the number of half-lives is $k = \frac{t}{h} = \frac{75}{25} = 3$, and the amount of material remaining is equal to $A_0\left(\frac{1}{2}\right)^3 = \frac{A_0}{8}$.

Radioactive carbon (carbon-14 or C-14), produced when nitrogen-14 is bombarded by cosmic rays in the atmosphere, drifts down to Earth and is absorbed from the air by plants. Animals eat the plants and take C-14 into their bodies. Humans in turn take C-14 into their bodies by eating both plants and animals. When a living organism dies, it stops absorbing C-14, and the C-14 that is already in the object begins to decay at a slow but steady rate, reverting to nitrogen-14. The half-life of C-14 is 5730 years. Half of the original amount of C-14 in the organic matter will have disintegrated after 5730 years; half of the remaining C-14 will have been lost after another 5730 years, and so forth. By measuring the ratio of C-14 to N-14, archaeologists are able to date organic materials. However, after about 50 000 years, the amount of C-14 remaining will be so small that the organic material cannot be dated reliably.

Half-life formula

If a certain initial amount, A_0, of material decays with a half-life of h, the amount of material that remains at time t is given by the exponential decay model $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$. The time units (e.g. seconds, hours, years) for h and t must be the same.

Example 4

The half-life of radioactive carbon-14 is approximately 5730 years. How much of a 10 g sample of carbon-14 remains after 15 000 years?

Solution

The exponential decay model for the carbon-14 is $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{t}{5730}}$. What remains of 10 g after 15 000 years is given by $A(15\,000) = 10\left(\frac{1}{2}\right)^{\frac{15\,000}{5730}} \approx 1.63$ g.

Compound interest

Recall from Chapter 3 that exponential functions occur in calculating compound interest. If an initial amount of money P, called the **principal**, is invested at an interest rate r per time period, then after one time period the amount of interest is $P \times r$ and the total amount of money is $A = P + Pr = P(1 + r)$. If the interest is added to the principal, the new principal is $P(1 + r)$, and the total amount after another time period is $A = P(1 + r)(1 + r) = P(1 + r)^2$. In the same way, after a third time period the amount is $A = P(1 + r)^3$. In general, after k periods the total amount is $A = P(1 + r)^k$, an exponential function with growth factor $1 + r$. For example, if the amount of money in a bank account is earning interest at a rate of 6.5% per time period, the growth factor is $1 + 0.065 = 1.065$. Is it possible for r to be negative? Yes, if an amount (not just money) is decreasing. For example, if the population of a town is decreasing by 12% per time period, the decay factor is $1 - 0.12 = 0.88$.

For compound interest, if the annual interest rate is r and interest is compounded (number of times added in) n times per year, then each time period the interest rate is $\frac{r}{n}$, and there are $n \times t$ time periods in t years.

Compound interest formula

The exponential function for calculating the amount of money after t years, $A(t)$, where P is the initial amount or principal, the annual interest rate is r and the number of times interest is compounded per year is n, is given by

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

Example 5

An initial amount of 1000 euros is deposited into an account earning $5\frac{1}{4}\%$ interest per year. Find the amounts in the account after eight years if interest is compounded annually, semi-annually, quarterly, monthly and daily.

Solution

We use the exponential function associated with compound interest with values of $P = 1000$, $r = 0.0525$ and $t = 8$.

Compounding	n	Amount after 8 years
Annual	1	$1000\left(1 + \frac{0.0525}{1}\right)^{8} = 1505.83$
Semi-annual	2	$1000\left(1 + \frac{0.0525}{2}\right)^{2(8)} = 1513.74$
Quarterly	4	$1000\left(1 + \frac{0.0525}{4}\right)^{4(8)} = 1517.81$
Monthly	12	$1000\left(1 + \frac{0.0525}{12}\right)^{12(8)} = 1520.57$
Daily	365	$1000\left(1 + \frac{0.0525}{365}\right)^{365(8)} = 1521.92$

Table 4.1

Example 6

A new car is purchased for \$22 000. If the value of the car decreases (depreciates) at a rate of approximately 15% per year, what will be the approximate value of the car to the nearest whole dollar in $4\frac{1}{2}$ years?

Solution

The decay rate for the exponential function is $1 - r = 1 - 0.15 = 0.85$. In other words, after each year the car's value is 85% of what it was one year before. We use the exponential decay model $A(t) = A_0b^t$ with values $A_0 = 22\,000$, $b = 0.85$ and $t = 4.5$.

$$A(4.5) = 22\,000(0.85)^{4.5} \approx 10\,588$$

The value of the car will be approximately \$10 588.

Exercise 4.1 and 4.2

For questions 1–3, sketch a graph of the function and state its domain, range, y-intercept and the equation of its horizontal asymptote.

1 $f(x) = 3^{x+4}$ **2** $g(x) = -2^x + 8$ **3** $h(x) = 4^{-x} - 1$

4 If a general exponential function is written in the form $f(x) = a(b)^{x-c} + d$, state the domain, range, y-intercept and the equation of the horizontal asymptote in terms of the parameters a, b, c and d.

5 Using your GDC and a graph-viewing window with Xmin = −2, Xmax = 2, Ymin = 0 and Ymax = 4, sketch a graph for each exponential equation on the same set of axes.
a) $y = 2^x$ b) $y = 4^x$ c) $y = 8^x$
d) $y = 2^{-x}$ e) $y = 4^{-x}$ f) $y = 8^{-x}$

6 Write equations that are equivalent to the equations in 5 d), e) and f) but have an exponent of positive x rather than negative x.

7 If $1 < a < b$, which is steeper: the graph of $y = a^x$ or $y = b^x$?

8 The population of a city triples every 25 years. At time $t = 0$, the population is 100 000. Write a function for the population $P(t)$ as a function of t. What is the population after:
a) 50 years b) 70 years c) 100 years?

9 An experiment involves a colony of bacteria in a solution. It is determined that the number of bacteria doubles approximately every 3 minutes and the initial number of bacteria at the start of the experiment is 10^4. Write a function for the number of bacteria $N(t)$ as a function of t (in minutes). Approximately how many bacteria are there after:
a) 3 minutes b) 9 minutes c) 27 minutes d) one hour?

10 If \$10 000 is invested at an annual interest rate of 11%, compounded quarterly, find the value of the investment after the given number of years.
a) 5 years b) 10 years c) 15 years

11 A sum of \$5000 is deposited into an investment account that earns interest at a rate of 9% per year compounded monthly.
a) Write the function $A(t)$ that computes the value of the investment after t years.
b) Use your GDC to sketch a graph of $A(t)$ with values of t on the horizontal axis ranging from $t = 0$ years to $t = 25$ years.
c) Use the graph on your GDC to determine the minimum number of years (to the nearest whole year) for this investment to have a value greater than \$20 000.

12 If \$10 000 is invested at an annual interest rate of 11% for a period of five years, find the value of the investment for the following compounding periods.
a) annually b) monthly c) daily d) hourly

13 Imagine a bank account that has the fantastic annual interest rate of 100%. If you deposit \$1 into this account, how much will be in the account exactly one year later, for the following compounding periods?
a) annually b) monthly c) daily d) hourly e) every minute

14 Each year for the past eight years, the population of deer in a national park increases at a steady rate of 3.2% per year. The present population is approximately 248 000.
a) What was the approximate number of deer one year ago?
b) What was the approximate number of deer eight years ago?

15 Radioactive carbon-14 has a half-life of 5730 years. The remains of an animal are found 20 000 years after it died. About what percentage (to 3 significant figures) of the original amount of carbon-14 (when the animal was alive) would you expect to find?

16 Once a certain drug enters the bloodstream of a human patient, it has a half-life of 36 hours. An amount of the drug, A_0, is injected in the bloodstream at 12:00 on Monday. How much of the drug will be in the bloodstream of the patient five days later at 12:00 on Friday?

17 Why are exponential functions of the form $f(x) = b^x$ defined so that $b > 0$?

18 You are offered a highly paid job that lasts for just one month – exactly 30 days. Which of the following payment plans, I or II, would give you the largest salary? How much would you get paid?
I One dollar on the first day of the month, two dollars on the second day, three dollars on the third day, and so on (getting paid one dollar more each day) until the end of the 30 days. (You would have a total of \$55 after 10 days.)
II One cent (\$0.01) on the first day of the month, two cents (\$0.02) on the second day, four cents on the third day, eight cents on the fourth day, and so on (each day getting paid double from the previous day) until the end of the 30 days. (You would have a total of \$10.23 after 10 days.)

4.3 The number e

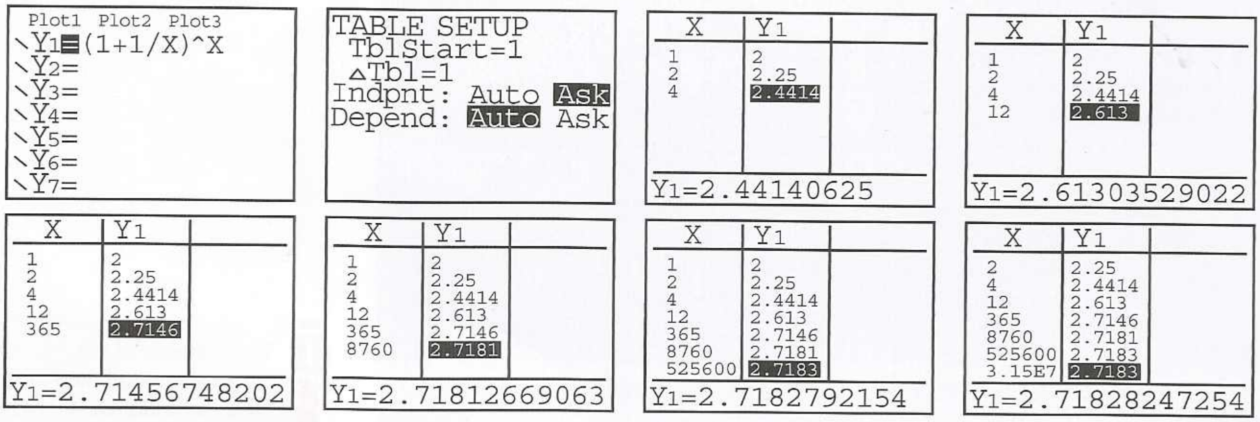
Recalling the definition of an exponential function $f(x) = b^x$, we recognize that any positive number can be used as the base b. Given that our number system is a base 10 system and that a base 2 number system (binary numbers) has useful applications (e.g. computers), it is understandable that exponential functions with base 2 or 10 are commonly used for modelling certain applications. However, the most important base is an irrational number that is denoted with the letter e. The value of e, approximated to 5 significant figures, is 2.71 828. The importance of e will be clearer when we get to calculus topics. The number π – another very useful irrational number – has a natural geometric significance as the ratio of circumference to diameter for any circle. Although not geometric, the number e also occurs in a 'natural' manner. We can see this by revisiting compound interest and considering **continuous change** rather than **incremental change**.

The 'discovery' of the constant e is attributed to Jakob Bernoulli (1654–1705). He was a member of the famous Bernoulli family of distinguished mathematicians, scientists and philosophers. This included his brother Johann (1667–1748), who made important developments in calculus, and his nephew Daniel (1700–1782), who is most well known for Bernoulli's principle in physics. The constant e is of enormous mathematical significance – and it appears 'naturally' in many mathematical processes. Jakob Bernoulli first observed e when studying sequences of numbers in connection to compound interest problems.

Continuously compounded interest

In the previous section and in Chapter 3, we computed amounts of money resulting from an initial amount (principal) with interest being compounded (added in) at discrete intervals (e.g. yearly, monthly, daily). In the formula that we used, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, n is the number of times that interest is compounded per year. Instead of adding interest only at discrete intervals, let's investigate what happens if we try to add interest continuously – that is, let the value of n increase without bound ($n \to \infty$).

Consider investing just \$1 at a very generous annual interest rate of 100%. How much will be in the account at the end of just one year? It depends on how often the interest is compounded. If it is only added at the end of the year ($n = 1$), the account will have \$2 at the end of the year. Is it possible to compound the interest more often to get a one-year balance of \$2.50 or of \$3.00? We use the compound interest formula with $P = \$1$, $r = 1.00$ (100%) and $t = 1$, and compute the amounts for increasing values of n. $A(1) = 1\left(1 + \frac{1}{n}\right)^{n \cdot 1} = \left(1 + \frac{1}{n}\right)^{n}$. This can be done very efficiently on your GDC by entering the equation $y = \left(1 + \frac{1}{x}\right)^{x}$ to display a table showing function values of increasing values of x.



As the number of compounding periods during the year increases, the amount at the end of the year appears to approach a limiting value.

As $n \to \infty$, the quantity of $\left(1 + \frac{1}{n}\right)^{n}$ approaches the number e. To 13 decimal places, e is approximately 2.718 281 828 4590.

Table 4.2

Compounding	n	$A(1) = \left(1 + \frac{1}{n}\right)^{n}$
Annual	1	2
Semi-annual	2	2.25
Quarterly	4	2.441 406 25…
Monthly	12	2.613 035 290 22…
Daily	365	2.714 567 482 02…
Hourly	8 760	2.718 126 690 63…
Every minute	525 600	2.718 279 2154…
Every second	31 536 000	2.718 282 472 54…

Leonhard Euler (1701–1783) was the dominant mathematical figure of the 18th century and is one of the most influential and prolific mathematicians of all time. Euler's collected works fill over 70 large volumes. Nearly every branch of mathematics has significant theorems that are attributed to Euler.

Euler proved mathematically that the limit of $\left(1 + \frac{1}{n}\right)^n$ as n goes to infinity is precisely equal to an irrational constant which he labelled e. His mathematical writings were influential not just because of the content and quantity but also because of Euler's insistence on clarity and efficient mathematical notation. Euler introduced many of the common algebraic notations that we use today. Along with the symbol e for the base of natural logarithms (1727), Euler introduced $f(x)$ for a function (1734), i for the square root of negative one (1777), π for pi, Σ for summation (1755), and many others. His introductory algebra text, written originally in German (Euler was Swiss), is still available in English translation. Euler spent most of his working life in Russia and Germany. Switzerland honoured Euler by placing his image on the 10 Swiss franc banknote.

Definition of *e*

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

The definition is read as 'e equals the limit of $\left(1 + \frac{1}{n}\right)^n$ as n goes to infinity'.

As the number of compoundings, n, increase without bound, we approach continuous compounding – where interest is being added continuously. In the formula for calculating amounts resulting from compound interest, letting $m = \frac{n}{r}$ produces

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

Now if $n \to \infty$ and the interest rate r is constant, then $\frac{n}{r} = m \to \infty$. From the limit definition of e, we know that if $m \to \infty$, then $\left(1 + \frac{1}{m}\right)^m \to e$.

Therefore, for continuous compounding, it follows that

$A(t) = P\left[\left(1 + \frac{1}{m}\right)^{n}\right]^{rt} = P[e]^{rt}$.

This result is part of the reason that e is the best choice for the base of an exponential function modelling change that occurs continually (e.g. radioactive decay) rather than in discrete intervals.

Continuous compound interest formula

The exponential function for calculating the amount of money after t years, $A(t)$, for interest compounded continuously, where P is the initial amount or principal and r is the annual interest rate, is given by $A(t) = Pe^{rt}$.

Example 7

An initial investment of 1000 euros earns interest at an annual rate of $7\frac{1}{2}\%$. Find the total amount after five years if the interest is compounded a) quarterly, and b) continuously.

Solution

a) $A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = 1000\left(1 + \frac{0.075}{4}\right)^{4 \cdot 5} = 1449.95$ euros

b) $A(t) = Pe^{rt} = 1000e^{0.075(5)} = 1454.99$ euros

The natural exponential function and continuous change

For many applications involving continuous change, the most suitable choice for a mathematical model is an exponential function with a base having the value of *e*.

The natural exponential function

The natural exponential function is the function defined as

$$f(x) = e^x$$

As with other exponential functions, the domain of the natural exponential function is the set of all real numbers ($x \in \mathbb{R}$), and its range is the set of positive real numbers ($y > 0$). The natural exponential function is often referred to as *the* exponential function.

The formula developed for continuously compounded interest does not apply only to applications involving adding interest to financial accounts. It can be used to model growth or decay of a quantity that is changing *geometrically* (i.e. repeated multiplication by a constant ratio, or growth/decay factor) and the change is continuous, or approaching continuous. Another version of a formula for continuous change, which we will learn more about in calculus, follows.

Continuous exponential growth/decay

If an initial quantity C (when $t = 0$) grows or decays continuously at a rate r over a certain time period, the amount $A(t)$ after t time periods is given by the function $A(t) = Ce^{rt}$.
If $r > 0$, the quantity is growing. If $r < 0$, the quantity is decreasing (decaying).

Example 8

A programme to reduce the number of rabbits has been taking place in a certain Australian city park. At the start of the programme there were 230 rabbits. After t years the number of rabbits, R, is modelled by $R = 230e^{-0.2t}$. How many rabbits are there after three years?

Solution

$R = 230e^{-0.2(3)} \approx 126.2$. There are approximately 126 rabbits after three years of the programme.

Exercise 4.3

1 Use your GDC to graph the curve $y = \left(1 + \frac{1}{x}\right)^x$ and the horizontal line $y = 2.72$. Use a graph window so that x ranges from 0 to 20 and y ranges from 0 to 3. Describe the behaviour of the graph of $y = \left(1 + \frac{1}{x}\right)^x$. Will it ever intersect the graph of $y = 2.72$? Explain.

2 Two different banks, Bank A and Bank B, offer accounts with exactly the same annual interest rate of 6.85%. However, the account from Bank A has the interest compounded monthly whereas the account from Bank B compounds the interest continuously. To decide which bank to open an account with, you calculate the **amount of interest** you would earn after three years from an initial deposit of 500 euros in each bank's account. It is assumed that you make no further deposits and no withdrawals during the three years. How much interest would you earn from each of the accounts? Which bank's account earns more – and how much more?

3 Dina wishes to deposit $1000 into an investment account and then withdraw the total in the account in five years. She has the choice of two different accounts. *Blue Star account*: interest is earned at an annual interest rate of 6.13% compounded weekly (52 weeks in a year). *Red Star account*: interest is earned at an annual interest rate of 5.95% compounded continuously. Which investment account – *Blue Star* **or** *Red Star* – will result in the greatest total at the end of five years? What is the total after five years for this account? How much more is it than the total for the other account?

4 Strontium-90 is a radioactive isotope of strontium. Strontium-90 decays according to the function $A(t) = Ce^{-0.0239t}$, where t is time in years and C is the initial amount of strontium-90 when $t = 0$. If you have 1 kilogram of strontium-90 to start with, how much (approximated to 3 significant figures) will you have after:

a) 1 year?

b) 10 years?

c) 100 years?

d) 250 years?

5 A radioactive substance decays in such a way that the mass (in kilograms) remaining after t days is given by the function $A(t) = 5e^{-0.0347t}$.

a) Find the mass (i.e. initial mass) at time $t = 0$.

b) What **percentage** of the initial mass remains after 10 days?

c) On your GDC and then on paper, draw a graph of the function $A(t)$ for $0 \leqslant t \leqslant 50$.

d) Use one of your graphs to approximate, to the nearest whole day, the half-life of the radioactive substance.

6 Which of the given interest rates and compounding periods would provide the better investment?

a) $8\frac{1}{2}$% per year, compounded semi-annually

b) $8\frac{1}{4}$% per year, compounded quarterly

c) 8% per year, compounded continuously

4.4 Logarithmic functions

In Example 7 of the previous section, we used the equation $A(t) = 1000e^{0.075t}$ to compute the amount of money in an account after t years. Now suppose we wish to determine how much time, t, it takes for the initial investment of 1000 euros to double. To find this we need to solve the following equation for t: $2000 = 1000e^{0.075t} \Rightarrow 2 = e^{0.075t}$. The unknown t is in the exponent. At this point in the book, we do not have an algebraic method to solve such an equation, but developing the concept of a **logarithm** will provide us with the means to do so.

John Napier (1550–1617) was a Scottish landowner, scholar and mathematician who 'invented' logarithms – a word he coined which derives from two Greek words: *logos* – meaning ratio, and *arithmos* – meaning number. Logarithms made numerical calculations much easier in areas such as astronomy, navigation, engineering and warfare. English mathematician Henry Briggs (1561–1630) came to Scotland to work with Napier and together they perfected logarithms, which included the idea of using the base ten. After Napier died in 1617, Briggs took over the work on logarithms and published a book of tables in 1624. By the second half of the 17th century, the use of logarithms had spread around the world. They became as popular as electronic calculators in our time. The great French mathematician Pierre-Simon Laplace (1749–1827) even suggested that the logarithms of Napier and Briggs doubled the life of astronomers, because it so greatly reduced the labours of calculation. In fact, without the invention of logarithms it is difficult to imagine how Kepler and Newton could have made their great scientific advances. In 1621, an English mathematician and clergyman, William Oughtred (1574–1660) used logarithms as the basis for the invention of the slide rule. The slide rule was a very effective calculation tool that remained in common use for over three hundred years.

The inverse of an exponential function

For $b > 1$, an exponential function with base b is increasing for all x, and for $0 < b < 1$ an exponential function is decreasing for all x. It follows from this that all exponential functions must be one-to-one. Recall from Section 2.3 that a one-to-one function passes both a vertical line test and a horizontal line test. We demonstrated that an inverse function would exist for any one-to-one function. Therefore, an exponential function with base b such that $b > 0$ and $b \neq 1$ will have an inverse function, which is given in the following definition. Also recall from Section 2.3 that the domain of a function $f(x)$ is the range of its inverse function $f^{-1}(x)$, and, similarly, the range of $f(x)$ is the domain of $f^{-1}(x)$. The domain and range are switched around for a function and its inverse.

Definition of a logarithmic function

For $b > 0$ and $b \neq 1$, the **logarithmic function** $y = \log_b x$ (read as 'logarithm with base b of x') is the inverse of the exponential function with base b.

$$y = \log_b x \text{ if and only if } x = b^y$$

The domain of the logarithmic function $y = \log_b x$ is the set of positive real numbers ($x > 0$) and its range is all real numbers ($y \in \mathbb{R}$).

Logarithmic expressions and equations

When evaluating logarithms, note that *a logarithm is an exponent*. This means that the value of $\log_b x$ is the exponent to which b must be raised to obtain x. For example, $\log_2 8 = 3$ because 2 must be raised to the power of 3 to obtain 8 – that is, $\log_2 8 = 3$ if and only if $2^3 = 8$.

We can use the definition of a logarithmic function to translate a logarithmic equation into an exponential equation and vice versa. When doing this, it is helpful to remember, as the definition stated, that in either form – logarithmic or exponential – the base is the same.

logarithmic equation — exponent ↓ $y = \log_b(x)$ ↑ base

exponential equation — exponent ↓ $x = b^y$ ↑ base

Example 9

Find the value of each of the following logarithms.

a) $\log_7 49$ b) $\log_5(\frac{1}{5})$ c) $\log_6 \sqrt{6}$ d) $\log_4 64$ e) $\log_{10} 0.001$

Solution

For each logarithmic expression in a) to e), we set it equal to y and use the definition of a logarithmic function to obtain an equivalent equation in exponential form. We then solve for y by applying the logical fact that if $b > 0, b \neq 1$ and $b^y = b^k$ then $y = k$.

a) Let $y = \log_7 49$ which is equivalent to the exponential equation $7^y = 49$. Since $49 = 7^2$, then $7^y = 7^2$. Therefore, $y = 2 \Rightarrow \log_7 49 = 2$.

b) Let $y = \log_5(\frac{1}{5})$ which is equivalent to the exponential equation $5^y = \frac{1}{5}$. Since $\frac{1}{5} = 5^{-1}$, then $5^y = 5^{-1}$. Therefore, $y = -1 \Rightarrow \log_5(\frac{1}{5}) = -1$.

c) Let $y = \log_6 \sqrt{6}$ which is equivalent to the exponential equation $6^y = \sqrt{6}$. Since $\sqrt{6} = 6^{\frac{1}{2}}$, then $6^y = 6^{\frac{1}{2}}$. Therefore, $y = \frac{1}{2} \Rightarrow \log_6 \sqrt{6} = \frac{1}{2}$.

d) Let $y = \log_4 64$ which is equivalent to the exponential equation $4^y = 64$. Since $64 = 4^3$, then $4^y = 4^3$. Therefore, $y = 3 \Rightarrow \log_4 64 = 3$.

e) Let $y = \log_{10} 0.001$ which is equivalent to the exponential equation $10^y = 0.001$. Since $0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}$, then $10^y = 10^{-3}$. Therefore, $y = -3 \Rightarrow \log_{10} 0.001 = -3$.

Example 10

Find the domain of the function $f(x) = \log_2(4 - x^2)$.

Solution

From the definition of a logarithmic function the domain of $y = \log_b x$ is $x > 0$, thus for $f(x)$ it follows that $4 - x^2 > 0 \Rightarrow (2 + x)(2 - x) > 0 \Rightarrow -2 < x < 2$.
Hence, the domain is $-2 < x < 2$.

Properties of logarithms

As with all functions and their inverses, their graphs are reflections of each other over the line $y = x$. Figure 4.4 illustrates this relationship for exponential and logarithmic functions, and also confirms the domain and range for the logarithmic function stated in the previous definition.

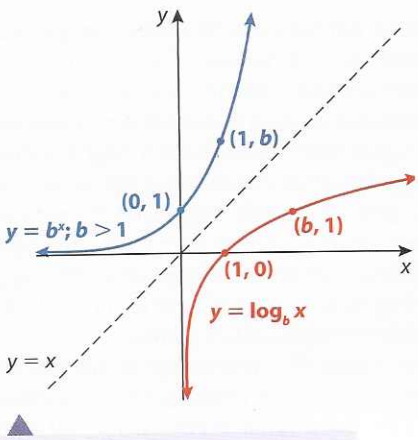


Figure 4.4

Notice that the points $(0, 1)$ and $(1, 0)$ are mirror images of each other over the line $y = x$. This corresponds to the fact that since $b^0 = 1$ then $\log_b 1 = 0$. Another pair of mirror image points, $(1, b)$ and $(b, 1)$, highlight the fact that $\log_b b = 1$.

Notice also that since the x-axis is a horizontal asymptote of $y = b^x$, the y-axis is a vertical asymptote of $y = \log_b x$.

In Section 2.3, we established that a function f and its inverse function f^{-1} satisfy the equations

$$f^{-1}(f(x)) = x \qquad \text{for } x \text{ in the domain of } f$$
$$f(f^{-1}(x)) = x \qquad \text{for } x \text{ in the domain of } f^{-1}$$

When applied to $f(x) = b^x$ and $f^{-1}(x) = \log_b x$, these equations become

$$\log_b(b^x) = x \qquad x \in \mathbb{R}$$
$$b^{\log_b x} = x \qquad x > 0$$

Properties of logarithms I

For $b > 0$ and $b \neq 1$, the following statements are true:

1. $\log_b 1 = 0$ (*because* $b^0 = 1$)
2. $\log_b b = 1$ (*because* $b^1 = b$)
3. $\log_b(b^x) = x$ (*because* $b^x = b^x$)
4. $b^{\log_b x} = x$ (*because* $\log_b x$ *is the power to which b must be raised to get x*)

The logarithmic function with base 10 is called the **common logarithmic function**. On calculators and on your GDC, this function is denoted by **log**. The value of the expression $\log_{10} 1000$ is 3 because 10^3 is 1000. Generally, for common logarithms (i.e. base 10) we omit writing the base of 10. Hence, if **log** is written with no base indicated, it is assumed to have a base of 10. For example, $\log 0.01 = -2$.

Common logarithm: $\log_{10} x = \log x$

As with exponential functions, the most widely used logarithmic function – and the other logarithmic function supplied on all calculators – is the logarithmic function with the base of e. This function is known as the **natural logarithmic function** and it is the inverse of the natural exponential function $y = e^x$. The natural logarithmic function is denoted by the symbol **ln**, and the expression $\ln x$ is read as 'the natural logarithm of x'.

Natural logarithm: $\log_e x = \ln x$

Example 11

Evaluate the following expressions:

a) $\log\left(\frac{1}{10}\right)$ b) $\log(\sqrt{10})$ c) $\log 1$ d) $10^{\log 47}$ e) $\log 50$

f) $\ln e$ g) $\ln\left(\frac{1}{e^3}\right)$ h) $\ln 1$ i) $e^{\ln 5}$ j) $\ln 50$

Solution

a) $\log\left(\frac{1}{10}\right) = \log(10^{-1}) = -1$ b) $\log(\sqrt{10}) = \log(10^{\frac{1}{2}}) = \frac{1}{2}$

c) $\log 1 = \log(10^0) = 0$ d) $10^{\log 47} = 47$

e) $\log 50 \approx 1.699$ (using GDC) f) $\ln e = 1$

g) $\ln\left(\frac{1}{e^3}\right) = \ln(e^{-3}) = -3$ h) $\ln 1 = \ln(e^0) = 0$

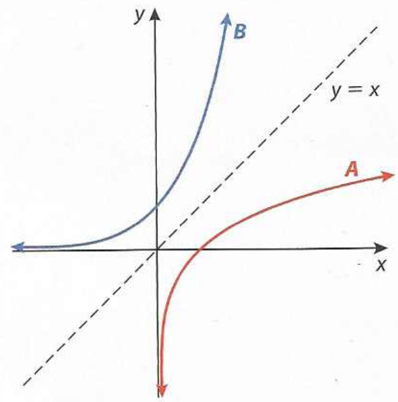
i) $e^{\ln 5} = 5$ j) $\ln 50 \approx 3.912$ (using GDC)

Example 12

The diagram shows the graph of the line $y = x$ and two curves. Curve A is the graph of the equation $y = \log x$. Curve B is the reflection of curve A in the line $y = x$.

a) Write the equation for curve B.

b) Write the coordinates of the y-intercept of curve B.



Solution

a) Curve A is the graph of $y = \log x$, the common logarithm with base 10, which could also be written as $y = \log_{10} x$. Curve B is the inverse of $y = \log_{10} x$, since it is the reflection of it in the line $y = x$. Hence, the equation for curve B is the exponential equation $y = 10^x$.

b) The y-intercept occurs when $x = 0$. For curve B, $y = 10^0 = 1$. Therefore, the y-intercept for curve B is $(0, 1)$.

The logarithmic function with base b is the inverse of the exponential function with base b. Therefore, it makes sense that the laws of exponents (Section 1.3) should have corresponding properties involving logarithms. For example, the exponential property $b^0 = 1$ corresponds to the logarithmic property $\log_b 1 = 0$. We will state and prove three further important logarithmic properties that correspond to the following three exponential properties.

1. $b^m \cdot b^n = b^{m+n}$
2. $\frac{b^m}{b^n} = b^{m-n}$
3. $(b^m)^n = b^{mn}$

Properties of logarithms II

Given $M > 0$, $N > 0$ and k is any real number, the following properties are true for logarithms with $b > 0$ and $b \neq 1$.

Property	Description
1. $\log_b(MN) = \log_b M + \log_b N$	the log of a product is the sum of the logs of its factors
2. $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$	the log of a quotient is the log of the numerator minus the log of the denominator
3. $\log_b(M^k) = k \log_b M$	the log of a number raised to an exponent is the exponent times the log of the number

Any of these properties can be applied in either direction.

Proofs

Property 1: Let $x = \log_b M$ and $y = \log_b N$.

The corresponding exponential forms of these two equations are

$$b^x = M \text{ and } b^y = N$$

Then, $\log_b(MN) = \log_b(b^x b^y) = \log_b(b^{x+y}) = x + y.$

It's given that $x = \log_b M$ and $y = \log_b N$; hence, $x + y = \log_b M + \log_b N.$

Therefore, $\log_b(MN) = \log_b M + \log_b N.$

Property 2: Again, let $x = \log_b M$ and $y = \log_b N \Rightarrow b^x = M$ and $b^y = N$.

Then, $\log_b\left(\frac{M}{N}\right) = \log_b\left(\frac{b^x}{b^y}\right) = \log_b(b^{x-y}) = x - y.$

With $x = \log_b M$ and $y = \log_b N$, then $x - y = \log_b M - \log_b N.$

Therefore, $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N.$

Property 3: Let $x = \log_b M \Rightarrow b^x = M.$

Now, let's take the logarithm of M^k and substitute b^x for M:

$\log_b(M^k) = \log_b[(b^x)^k] = \log_b(b^{kx}) = kx$

It's given that $x = \log_b M$; hence, $kx = k\log_b M.$

Therefore, $\log_b(M^k) = k\log_b M.$

• **Hint:** The notation $f(x)$ uses brackets *not* to indicate multiplication but to indicate the argument of the function f. The symbol f is the name of a function, not a variable – it is not multiplying the variable x. Therefore, $f(x + y)$ is NOT equal to $f(x) + f(y)$. Likewise, the symbol **log** is also the name of a function. Therefore, $\log_b(x + y)$ is not equal to $\log_b(x) + \log_b(y)$. Other mistakes to avoid include incorrectly simplifying quotients or powers of logarithms. Specifically, $\frac{\log_b x}{\log_b y} \neq \log\left(\frac{x}{y}\right)$ and $(\log_b x)^k \neq k(\log_b x)$.

Example 13

Use the properties of logarithms to write each logarithmic expression as a sum, difference, and/or constant multiple of simple logarithms (i.e. logarithms without sums, products, quotients or exponents).

a) $\log_2(8x)$ b) $\ln\left(\frac{3}{y}\right)$ c) $\log(\sqrt{7})$

d) $\log_b\left(\frac{x^3}{y^2}\right)$ e) $\ln(5e^2)$ f) $\log\left(\frac{m+n}{n}\right)$

Solution

a) $\log_2(8x) = \log_2 8 + \log_2 x = 3 + \log_2 x$

b) $\ln\left(\frac{3}{y}\right) = \ln 3 - \ln y$

c) $\log(\sqrt{7}) = \log(7^{\frac{1}{2}}) = \frac{1}{2}\log 7$

d) $\log_b\left(\frac{x^3}{y^2}\right) = \log_b(x^3) - \log_b(y^2) = 3\log_b x - 2\log_b y$

e) $\ln(5e^2) = \ln 5 + \ln(e^2) = \ln 5 + 2\ln e = \ln 5 + 2(1) = 2 + \ln 5$
   $(2 + \ln 5 \approx 3.609$ using GDC)

f) $\log\left(\frac{m+n}{m}\right) = \log(m + n) - \log m$
(remember $\log(m + n) \neq \log m + \log n$)

Example 14

Write each expression as the logarithm of a single quantity.

a) $\log 6 + \log x$

b) $\log_2 5 + 2\log_2 3$

c) $\ln y - \ln 4$

d) $\log_b 12 - \frac{1}{2}\log_b 9$

e) $\log_3 M + \log_3 N - 2\log_3 P$

f) $\log_2 80 - \log_2 5$

Solution

a) $\log 6 + \log x = \log(6x)$

b) $\log_2 5 + 2\log_2 3 = \log_2 5 + \log_2(3^2) = \log_2 5 + \log_2 9 = \log_2(5 \cdot 9)$
$= \log_2 45$

c) $\ln y - \ln 4 = \ln\left(\frac{y}{4}\right)$

d) $\log_b 12 - \frac{1}{2}\log_b 9 = \log_b 12 - \log_b(9^{\frac{1}{2}}) = \log_b 12 - \log_b(\sqrt{9})$
$= \log_b 12 - \log_b 3 = \log_b\left(\frac{12}{3}\right) = \log_b 4$

e) $\log_3 M + \log_3 N - 2\log_3 P = \log_3(MN) - \log_3(P^2) = \log_3\left(\frac{MN}{P^2}\right)$

f) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right) = \log_2 16 = 4$ (because $2^4 = 16$)

Change of base

The answer to part f) of Example 14 was $\log_2 16$ which we can compute to be exactly 4 because we know that $2^4 = 16$. The answer to part e) of Example 13 was $2 + \ln 5$ which we approximated to 3.609 using the natural logarithm function key (**ln**) on our GDC. But, what if we wanted to compute an approximate value for $\log_2 45$, the answer to part b) of Example 14? Our GDC can only evaluate common logarithms (base 10) and natural logarithms (base e). To evaluate logarithmic expressions and graph logarithmic functions to other bases we need to apply a **change of base formula.**

Change of base formula

Let a, b and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_b x$ can be expressed in terms of logarithms to any other base a as follows:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

Proof

$y = \log_b x \Rightarrow b^y = x$	convert from logarithmic form to exponential form
$\log_a x = \log_a(b^y)$	if $b^y = x$, then log of each with same bases must be equal
$\log_a x = y\log_a b$	applying the property $\log_b(M^k) = k\log_b M$
$y = \frac{\log_a x}{\log_a b}$	divide both sides by $\log_a b$
$\log_b x = \frac{\log_a x}{\log_a b}$	substitute $\log_b x$ for y

To apply the change of base formula, let $a = 10$ or $a = e$. Then the logarithm of any base b can be expressed in terms of either common logarithms or natural logarithms. For example:

$$\log_2 x = \frac{\log x}{\log 2} \quad \text{or} \quad \frac{\ln x}{\ln 2}$$

$$\log_5 x = \frac{\log x}{\log 5} \quad \text{or} \quad \frac{\ln x}{\ln 5}$$

$$\log_2 45 = \frac{\log 45}{\log 2} = \frac{\ln 45}{\ln 2} \approx 5.492 \quad \text{(using GDC)}$$

Example 15

Use the change of base formula and common or natural logarithms to evaluate each logarithmic expression. Start by making a rough mental estimate. Approximate your answer to 4 significant figures.

a) $\log_3 30$

b) $\log_9 6$

Solution

a) The value of $\log_3 30$ is the power to which 3 is raised to obtain 30. Because $3^3 = 27$ and $3^4 = 81$, the value of $\log_3 30$ is between 3 and 4, and will be much closer to 3 than 4 – perhaps around 3.1. Using the change of base formula and common logarithms, we obtain $\log_3 30 = \frac{\log 30}{\log 3} \approx 3.096$. This agrees well with the mental estimate.

After computing the answer on your GDC, use your GDC to also check it by raising 3 to the answer and confirming that it gives a result of 30.

```
log(30)/log(3)
          3.095903274
3^Ans
                   30
```

b) The value of $\log_9 6$ is the power to which 9 is raised to obtain 6. Because $9^{\frac{1}{2}} = \sqrt{9} = 3$ and $9^1 = 9$, the value of $\log_9 6$ is between $\frac{1}{2}$ and 1 – perhaps around 0.75. Using the change of base formula and natural logarithms, we obtain $\log_9 6 = \frac{\ln 6}{\ln 9} \approx 0.815$. This agrees well with the mental estimate.

```
ln(6)/ln(9)
          .8154648768
9^Ans
                    6
```

Exercise 4.4

In questions 1–9, express each logarithmic equation as an exponential equation.

1 $\log_2 16 = 4$ **2** $\ln 1 = 0$ **3** $\log 100 = 2$

4 $\log 0.01 = -2$ **5** $\log_7 343 = 3$ **6** $\ln\left(\frac{1}{e}\right) = -1$

7 $\log 50 = y$ **8** $\ln x = 12$ **9** $\ln(x + 2) = 3$

In questions10–18, express each exponential equation as a logarithmic equation.

10 $2^{10} = 1024$ **11** $10^{-4} = 0.0001$ **12** $4^{-\frac{1}{2}} = \frac{1}{2}$

13 $3^4 = 81$ **14** $10^0 = 1$ **15** $e^x = 5$

16 $2^{-3} = 0.125$ **17** $e^4 = y$ **18** $10^{x+1} = y$

In questions 19–34, find the exact value of the expression without using your GDC.

19 $\log_2 64$ **20** $\log_4 64$ **21** $\log_2\left(\frac{1}{8}\right)$ **22** $\log_3(3^5)$

23 $\log_8 1$ **24** $10^{\log 6}$ **25** $\log_3\left(\frac{1}{27}\right)$ **26** $e^{\ln\sqrt{2}}$

27 $\log 1000$ **28** $\ln(\sqrt{e})$ **29** $\ln\left(\frac{1}{e^2}\right)$ **30** $\log 0.001$

31 $\log_4 2$ **32** $3^{\log_3 18}$ **33** $\log_5(\sqrt[3]{5})$ **34** $10^{\log \pi}$

In questions 35–42, use a GDC to evaluate the expression, correct to 4 significant figures.

35 $\log 50$ **36** $\log\sqrt{3}$ **37** $\ln 50$ **38** $\ln\sqrt{3}$

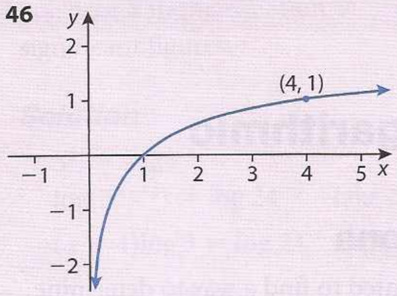
39 $\log 25$ **40** $\log\left(\frac{1+\sqrt{5}}{2}\right)$ **41** $\ln 100$ **42** $\ln(100^3)$

In questions 43–45, find the domain of the logarithmic function.

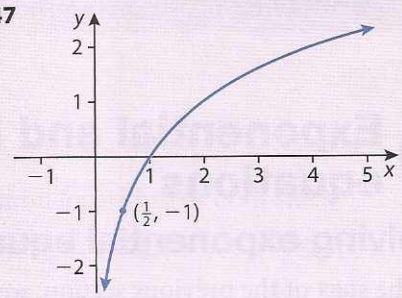
43 $f(x) = \log(x - 2)$ **44** $g(x) = \ln(x^2)$ **45** $h(x) = \log(x) - 2$

For questions 46–49, find the equation of the function that is graphed in the form $f(x) = \log_b x$.

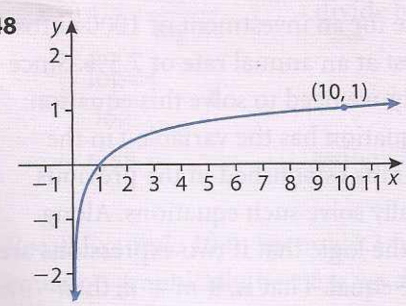
46



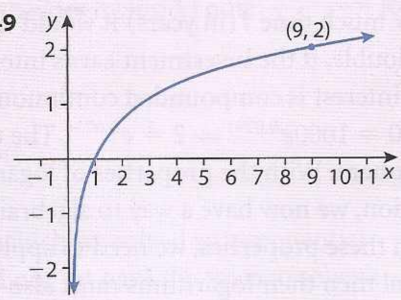
47



48



49



In questions 50–55, use properties of logarithms to write each logarithmic expression as a sum, difference and/or constant multiple of simple logarithms (i.e. logarithms without sums, products, quotients or exponents).

50 $\log_2(2m)$

51 $\log\left(\frac{9}{x}\right)$

52 $\ln(\sqrt[5]{x})$

53 $\log_3(ab^3)$

54 $\log[10x(1 + r)^t]$

55 $\ln\left(\frac{m^3}{n}\right)$

In questions 56–61, write each expression as the logarithm of a single quantity.

56 $\log(x^2) + \log\left(\frac{1}{x}\right)$

57 $\log_3 9 + 3\log_3 2$

58 $4\ln y - \ln 4$

59 $\log_b 12 - \frac{1}{2}\log_b 9$

60 $\log p - \log q - \log r$

61 $2\ln 6 - 1$ • **Hint:** $\ln(?) = 1$

In questions 62–65, use the change of base formula and common or natural logarithms to evaluate each logarithmic expression. Approximate your answer to 3 significant figures.

62 $\log_2 1000$

63 $\log_{\frac{1}{2}} 40$

64 $\log_6 40$

65 $\log_5(0.75)$

In questions 66 and 67, use the change of base formula to evaluate $f(20)$.

66 $f(x) = \log_2 x$

67 $f(x) = \log_5 x$

68 Use the change of base formula to prove the following statement.

$$\log_b a = \frac{1}{\log_a b}$$

69 Show that $\log e = \frac{1}{\ln 10}$.

70 The relationship between the number of decibels dB (one variable) and the intensity I of a sound (in watts per square metre) is given by the formula $dB = 10\log\left(\frac{I}{10^{-16}}\right)$. Use properties of logarithms to write the formula in simpler form. Then find the number of decibels of a sound with an intensity of 10^{-4} watts per square metre.

4.5 Exponential and logarithmic equations

Solving exponential equations

At the start of the previous section, we wanted to find a way to determine how much time t (in years) it would take for an investment of 1000 euros to double, if the investment earns interest at an annual rate of 7.5%. Since the interest is compounded continuously, we need to solve this equation: $2000 = 1000e^{0.075t} \Rightarrow 2 = e^{0.075t}$. The equation has the variable t in the exponent. With the properties of logarithms established in the previous section, we now have a way to algebraically solve such equations. Along with these properties, we need to apply the logic that if two expressions are equal then their logarithms must also be equal. That is, if $m = n$, then $\log_b m = \log_b n$.

Example 16

Solve the equation for the variable t. Give your answer accurate to 3 significant figures.

$$2 = e^{0.075t}$$

Solution

$$2 = e^{0.075t}$$

$\ln 2 = \ln(e^{0.075t})$ take natural logarithm of both sides

$\ln 2 = 0.075t$ apply the property $\log_b(b^x) = x$ and $\ln e = 1$

$$t = \frac{\ln 2}{0.075} \approx 9.24$$

With interest compounding continuously at an annual interest rate of 7.5%, it takes about 9.24 years for the investment to double.

This example serves to illustrate a general strategy for solving exponential equations. To solve an exponential equation, first isolate the exponential expression and take the logarithm of both sides. Then apply a property of logarithms so that the variable is no longer in the exponent and it can be isolated on one side of the equation. By taking the logarithm of both sides of an exponential equation, we are making use of the inverse relationship between exponential and logarithmic functions. Symbolically, this method can be represented as follows – solving for x:

(i) If $b = 10$ or e: $y = b^x \Rightarrow \log_b y = \log_b b^x \Rightarrow \log_b y = x$

(ii) If $b \neq 10$ or e:

$$y = b^x \Rightarrow \log_a y = \log_a b^x \Rightarrow \log_a y = x \log_a b \Rightarrow x = \frac{\log_a y}{\log_a b}$$

Example 17

Solve for x in the equation $3^{x-4} = 24$. Approximate the answer to 3 significant figures.

Solution

$$3^{x-4} = 24$$

$\log(3^{x-4}) = \log 24$ take common logarithm of both sides

$(x-4)\log 3 = \log 24$ apply the property $\log_b(M^k) = k \log_b M$

$x - 4 = \dfrac{\log 24}{\log 3}$ divide both sides by log 3 $\left[\text{note: } \dfrac{\log 24}{\log 3} \neq \log 8\right]$

$$x = \frac{\log 24}{\log 3} + 4$$

$x \approx 6.89$ using GDC

Recall Example 10 in Section 3.3 in which we solved an exponential equation graphically, because we did not yet have the tools to solve it algebraically. Let's solve it now using logarithms.

• **Hint:** We could have used natural logarithms instead of common logarithms to solve the equation in Example 17. Using the same method but with natural logarithms, we get

$$x = \frac{\ln 24}{\ln 3} + 4 \approx 6.89.$$

Example 18

You invested €1000 at 6% compounded quarterly. How long will it take this investment to increase to €2000?

• **Hint:** Be sure to use brackets appropriately when entering the expression $\frac{\ln 2}{4 \ln 1.015}$ on your GDC. Following the rules for order of operations, your GDC will give an incorrect result if entered as shown here.

```
ln(2)/(4ln(1.015))
       .0025799999
```

missing brackets

Solution

Using the compound interest formula from Section 4.2, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$, with $P =$ €1000, $r = 0.06$ and $n = 4$, we need to solve for t when $A(t) = 2P$.

$$2P = P\left(1 + \frac{0.06}{4}\right)^{4t} \quad \text{substitute } 2P \text{ for } A(t)$$

$$2 = 1.015^{4t} \quad \text{divide both sides by } P$$

$$\ln 2 = \ln(1.015^{4t}) \quad \text{take natural logarithm of both sides}$$

$$\ln 2 = 4t \ln 1.015 \quad \text{apply the property } \log_b(M^k) = k \log_b M$$

$$t = \frac{\ln 2}{4 \ln 1.015}$$

$$t \approx 11.6389 \quad \text{evaluated on GDC}$$

```
ln(2)/(4ln(1.015
))
         11.63888141
```

The investment will double in 11.64 years – about 11 years and 8 months.

Example 19

The bacteria that cause 'strep throat' will grow in number at a rate of about 2.3% per minute. To the nearest whole minute, how long will it take for these bacteria to double in number?

Solution

Let t represent time in minutes and let A_0 represent the number of bacteria at $t = 0$.

Using the exponential growth model from Section 4.2, $A(t) = A_0 b^t$, the growth factor, b, is $1 + 0.023 = 1.023$ giving $A(t) = A_0(1.023)^t$. The same equation would apply to money earning 2.3% annual interest with the money being added (compounded) once per year rather than once per minute. So, our mathematical model assumes that the number of bacteria increase incrementally, with the number increasing by 2.3% at the end of each minute. To find the doubling time, find the value of t so that $A(t) = 2A_0$.

$$2A_0 = A_0(1.023)^t \quad \text{substitute } 2A_0 \text{ for } A(t)$$

$$2 = 1.023^t \quad \text{divide both sides by } A_0$$

$$\ln 2 = \ln(1.023^t) \quad \text{take natural logarithm of both sides}$$

$$\ln 2 = t \ln 1.023 \quad \text{apply the property } \log_b(M^k) = k \log_b M$$

$$t = \frac{\ln 2}{\ln 1.023} \approx 30.482$$

The number of bacteria will double in about 30 minutes.

Alternative solution

What if we assumed continuous growth instead of incremental growth? We apply the continuous exponential growth model from Section 4.3: $A(t) = Ce^{rt}$ with initial amount C and $r = 0.023$.

$2C = Ce^{0.023t}$	substitute $2C$ for $A(t)$
$2 = e^{0.023t}$	divide both sides by C
$\ln 2 = \ln(e^{0.023t})$	take natural logarithm of both sides
$\ln 2 = 0.023t$	apply the property $\log_b(b^x) = x$
$t = \dfrac{\ln 2}{0.023} \approx 30.137$	

Continuous growth has a slightly shorter doubling time, but rounded to the nearest minute it also gives an answer of 30 minutes.

Example 20

\$1000 is invested in an investment account that earns interest at an annual rate of 10% compounded monthly. Calculate the minimum number of years needed for the amount in the account to exceed \$4000.

Solution

We use the exponential function associated with compound interest, $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$ with $P = 1000$, $r = 0.1$ and $n = 12$.

$$4000 = 1000\left(1 + \frac{0.1}{12}\right)^{12t} \Rightarrow 4 = (1.008\overline{3})^{12t} \Rightarrow \log 4 = \log\left[(1.008\overline{3})^{12t}\right] \Rightarrow$$

$$\log 4 = 12t\log(1.008\overline{3}) \Rightarrow t = \frac{\log 4}{12\log(1.008\overline{3})} \approx 13.92 \text{ years}$$

The minimum number of years needed for the account to exceed \$4000 is 14 years.

Example 21

A 20 g sample of radioactive iodine decays so that the mass remaining after t days is given by the equation $A(t) = 20e^{-0.087t}$, where $A(t)$ is measured in grams. After how many days (to the nearest whole day) is there only 5 g remaining?

Solution

$$5 = 20e^{-0.087t} \Rightarrow \frac{5}{20} = e^{-0.087t} \Rightarrow \ln 0.25 = \ln(e^{-0.087t}) \Rightarrow$$

$$\ln 0.25 = -0.087t \Rightarrow t = \frac{\ln 0.25}{-0.087} \approx 15.93$$

After about 16 days there is only 5 g remaining.

Solving logarithmic equations

A logarithmic equation is an equation where the variable appears within the argument of a logarithm. For example, $\log x = \frac{1}{2}$ or $\ln x = 4$. We can solve both of these logarithmic equations directly by applying the definition of a logarithmic function (Section 4.4):

$$y = \log_b x \text{ if and only if } x = b^y$$

The logarithmic equation $\log x = \frac{1}{2}$ is equivalent to the exponential equation $x = 10^{\frac{1}{2}} = \sqrt{10}$, which leads directly to the solution. Likewise, the equation $\ln x = 4$ is equivalent to $x = e^4 \approx 54.598$. Both of these equations could have been solved by means of another method that makes use of the following two facts:

$$\text{(i) if } a = b \text{ then } n^a = n^b; \quad \text{and (ii) } b^{\log_b x} = x$$

To understand (ii) above, remember that a **logarithm is an exponent**. The value of $\log_b x$ is the exponent to which b is raised to give x. And b is being raised to this value; hence, the expression $b^{\log_b x}$ is equivalent to x. Therefore, another method for solving the logarithmic equation $\ln x = 4$ is to **exponentiate** both sides, i.e. use the expressions on either side of the equal sign as exponents for exponential expressions with equal bases. The base needs to be the base of the logarithm.

$$\ln x = 4 \Rightarrow e^{\ln x} = e^4 \Rightarrow x = e^4$$

Example 22

Solve for x: $\log_3(2x - 5) = 2$

Solution

$$\log_3(2x - 5) = 2 \Rightarrow 3^{\log_3(2x-5)} = 3^2$$
$$2x - 5 = 9$$
$$2x = 14$$
$$x = 7$$

Example 23

Solve for x in terms of k: $\log_2(5x) = 3 + k$

Solution

$\log_2(5x) = 3 + k \Rightarrow 2^{\log_2(5x)} = 2^{3+k}$ — exponentiate both sides with base = 2

$5x = 2^3 \cdot 2^k$ — law of exponents $b^m \cdot b^n = b^{m+n}$ used 'in reverse'

$$x = \frac{8}{5}(2^k)$$

For some logarithmic equations, it is necessary to first apply a property, or properties, of logarithms to simplify combinations of logarithmic expressions before solving.

Example 24

Solve for x: $\log_2 x + \log_2(10 - x) = 4$

Solution

$\log_2 x + \log_2(10 - x) = 4$

$\log_2[x(10 - x)] = 4$ property of logarithms: $\log_b M + \log_b N = \log_b(MN)$

$10x - x^2 = 2^4$ changing from logarithmic form to exponential form

$x^2 - 10x + 16 = 0$

$(x - 2)(x - 8) = 0$

$x = 2$ or $x = 8$

When solving logarithmic equations, you should be careful to always check if the *original* equation is a true statement when any solutions are substituted in for the variable. For Example 24, both of the solutions $x = 2$ and $x = 8$ produce true statements when substituted into the original equations. Sometimes 'extra' (extraneous) invalid solutions are produced, as illustrated in the next example.

Example 25

Solve for x: $\ln(x - 2) + \ln(2x - 3) = 2\ln x$

Solution

$\ln(x - 2) + \ln(2x - 3) = 2\ln x$

$\ln[(x - 2)(2x - 3)] = \ln x^2$ properties of logarithms

$\ln(2x^2 - 7x + 6) = \ln x^2$

$e^{\ln(2x^2 - 7x + 6)} = e^{\ln x^2}$ exponentiate both sides

$2x^2 - 7x + 6 = x^2$

$x^2 - 7x + 6 = 0$

$(x - 6)(x - 1) = 0$ factorize

$x = 6$ or $x = 1$

Substituting these two *possible* solutions indicates that $x = 1$ is not a valid solution. The reason is that if you try to substitute 1 for x into the original equation, we are not able to evaluate the expression $\ln(2x - 3)$ because we can only take the logarithm of a positive number. Therefore, $x = 6$ is the only solution. $x = 1$ is an extraneous solution that is not valid.

Solving, or checking the solutions to, a logarithmic equation on your GDC will help you avoid, or determine, extraneous solutions. To solve Example 25 on your GDC, a useful approach is to first set the equation equal to zero. Then graph the expression (after setting it equal to y) and observe where the graph intersects the x-axis (i.e. $y = 0$).

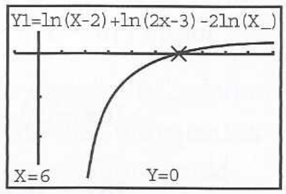
Graphical solution for Example 25:

$\ln(x-2)+\ln(2x-3)=2\ln x \Rightarrow \ln(x-2)+\ln(2x-3)-2\ln x=0$

Graph the equation $y=\ln(x-2)+\ln(2x-3)-2\ln x$ on your GDC and find x-intercepts.

```
Plot1 Plot2 Plot3
\Y1■ln(X-2)+ln(2
 X-3)-2ln(X)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=-1
 Xmax=10
 Xscl=1
 Ymin=-3
 Ymax=1
 Yscl=1
 Xres=1
```



The graph only intersects the x-axis at $x=6$ and not at $x=1$. Hence, $x=6$ is the only valid solution and $x=1$ is an extraneous solution.

Exercise 4.5

In questions 1–12, solve for x in the exponential equation. Give x accurate to 3 significant figures.

1 $10^x=5$ **2** $4^x=32$ **3** $8^{x-6}=60$

4 $2^{x+3}=100$ **5** $\left(\frac{1}{5}\right)^x=22$ **6** $e^x=15$

7 $10^x=e$ **8** $3^{2x-1}=35$ **9** $2^{x+1}=3^{x-1}$

10 $2e^{10x}=19$ **11** $6^{\frac{x}{2}}=5^{1-x}$ **12** $\left(1+\frac{0.05}{12}\right)^{12x}=3$

13 \$5000 is invested in an account that pays 7.5% interest per year, compounded quarterly.
a) Find the amount in the account after three years.
b) How long will it take for the money in the account to double? Give the answer to the nearest quarter of a year.

14 How long will it take for an investment of €500 to triple in value if the interest is 8.5% per year, compounded continuously. Give the answer in number of years accurate to 3 significant figures.

15 A single bacterium begins a colony in a laboratory dish. If the colony doubles every hour, after how many hours does the colony first have more than one million bacteria?

16 Find the least number of years for an investment to double if interest is compounded annually with the following interest rates.
a) 3% b) 6% c) 9%

17 A new car purchased in 2005 decreases in value by 11% per year. When is the first year that the car is worth less than one-half of its original value?

18 Uranium-235 is a radioactive substance that has a half-life of 2.7×10^5 years.
a) Find the amount remaining from a 1 g sample after a thousand years.
b) How long will it take a 1 g sample to decompose until its mass is 700 milligrams (i.e. 0.7 g)? Give the answer in years accurate to 3 significant figures.

19 The stray dog population in a town is growing exponentially with about 18% more stray dogs each year. In 2008, there are 16 stray dogs.

a) Find the projected population of stray dogs after five years.

b) When is the first year that the number of stray dogs is greater than 70?

20 Initially a water tank contains one thousand litres of water. At the time $t = 0$ minutes, a tap is opened and water flows out of the tank. The volume, V litres, which remains in the tank after t minutes is given by the following exponential function: $V(t) = 1000(0.925)^t$.

a) Find the value of V after 10 minutes.

b) Find how long, to the nearest second, it takes for half of the initial amount of water to flow out of the tank.

c) The tank is considered 'empty' when only 5% of the water remains. From when the tap is first opened, how many whole minutes have passed before the tank can first be considered empty?

21 The mass m kilograms of a radioactive substance at time t days is given by $m = 5e^{-0.13t}$.

a) What is the initial mass?

b) How long does it take for the substance to decay to 0.5 kg? Give the answer in days accurate to 3 significant figures.

In questions 22–32, solve for x in the logarithmic equation. Give exact answers and be sure to check for extraneous solutions.

22 $\log_2(3x - 4) = 4$

23 $\log(x - 4) = 2$

24 $\ln x = -3$

25 $\log_{16} x = \frac{1}{2}$

26 $\log \sqrt{x + 2} = 1$

27 $\ln(x^2) = 16$

28 $\log_2(x^2 + 8) = \log_2 x + \log_2 6$

29 $\log_3(x - 8) + \log_3 x = 2$

30 $\log 7 - \log(4x + 5) + \log(2x - 3) = 0$

31 $\log_3 x + \log_3(x - 2) = 1$

32 $\log x^8 = (\log x)^4$

Practice questions

1 Solve for x in each equation.

a) $\log_x 16 = 4$

b) $\log_3 27 = x$

c) $\log_8 x = -\frac{1}{3}$

d) $\log(x + 2) + \log(x - 2) = \log 5$

2 Solve for x in each equation.

a) $4^x = 36$

b) $5 \times 3^x = 18$

c) $8^{-x} = \left(\frac{1}{4}\right)^3$

d) $6^x = 0.25^{2x-1}$

3 Write each expression as the logarithm of a single quantity.

a) $\log_2 x^2 - \log_2 x + 2\log_2 3$

b) $\ln 3 + \frac{1}{2}\ln(x - 4) - \ln x$

4 If $\log_b M = 5.42$ and $\log_b N^2 = 3.78$, find the following:

a) $\log_b N$

b) $\log_b\left(\frac{N^4}{\sqrt{M}}\right)$

5 Pablo invested 2000 euros at an annual rate of 6.75%, compounded annually.

a) Find the value of Pablo's investment after four years. Give your answer to the nearest euro.

b) How many years will it take for Pablo's investment to double in value?

c) What should the interest rate be if Pablo's initial investment were to double in value in 10 years?

6 Let $\log P = x$, $\log Q = y$ and $\log R = z$.

Express $\log\left(\frac{P}{QR^3}\right)^2$ in terms of x, y and z.

7 \$1000 is deposited into a bank account that earns interest at an annual rate of 4% compounded annually. After three years, the annual interest rate is increased to 7% for a further four years.

a) How much money is in the account after the seven years?

b) Find what constant rate of annual interest compounded annually would have given the same amount of money in the seven years. Give your answer as a percentage to 1 decimal place.

8 Express each of the following expressions as simply as possible.

a) $\log_2 5 \times \log_5 2$

b) $\log_4 8$

c) $4^{\log_2 6}$

9 At the start of the year 2000 there were 500 elephants in a game reserve. After t years, the number of elephants E is given by $500(1.032)^t$.

a) Find the number of elephants at the start of 2006.

b) After how many full years will the number of elephants first become greater than 750?

10 The half-life of radioactive radium is 1620 years. What percentage of an initial amount of radioactive radium will remain after 100 years?

11 A car, when purchased new, had an initial value of \$25 000. After one year, the car had decreased in value to \$22 000.

a) After one year, what percentage of the initial value is the new value of the car?

b) If the car continues to decrease in value at the same annual rate, what is the car's value after six years? Give your answer to the nearest dollar.

c) If the car was purchased in 2002 in which year is the car first worth less than \$8000?

12 Consider the function $f: x \mapsto e^{x-2}$.

a) Write down the domain and range of f.

b) Write down the coordinates of any y-intercept, and the equation of any asymptotes for the graph of f.

c) Find f^{-1}.

d) Write down the domain and range of f^{-1}.

13 A population of a certain insect grows at a rate of 6% per month. Initially there are 500 insects.

a) Find the size of the population after four months.

b) Find the size of the population after sixteen months.

c) Let the size of the population after t months be given by the function $f(t) = A_0 b^t$. Write down

(i) the value A_0

(ii) the value of b.

An alternative way of modelling the size of the insect population is given by the function $g(t) = 500e^{kt}$.

d) By equating $f(t)$ and $g(t)$, find the value of k. Give your answer correct to 5 decimal places.

14 State the domain for each of the following two functions.

a) $f(x) = \log\left(\frac{x}{x-2}\right)$

b) $g(x) = \log x - \log(x - 2)$

Solve each of the following equations.

c) $\log\left(\frac{x}{x-2}\right) = -2$

d) $\log x - \log(x - 2) = -2$

15 An experiment is designed to study a certain type of bacteria. The number of bacteria after t minutes is given by an exponential function of the form $A(t) = Ce^{kt}$, where C and k are constants. At the start of the experiment (when $t = 0$) there are 5000 bacteria. After 22 minutes, the number of bacteria has increased to 17 000.

a) Find the exact value of C and an approximate value of k (to 3 significant figures).

b) How many bacteria does the exponential function predict there will be after one hour?