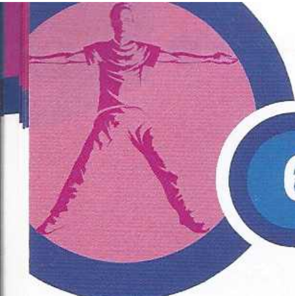


6 Trigonometric Functions and Equations

Assessment statements

3.1 The circle: radian measure of angles; length of an arc; area of a sector.

3.2 Definition of $\cos\theta$ and $\sin\theta$ in terms of the unit circle.
Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$.
The identity $\cos^2\theta + \sin^2\theta = 1$.

3.3 Double angle formulae: $\sin 2\theta = 2\sin\theta\cos\theta$;
$\cos 2\theta = \cos^2\theta - \sin^2\theta$.

3.4 The circular functions $\sin x$, $\cos x$ and $\tan x$: their domains and ranges; their periodic nature; and their graphs.
Composite functions of the form $f(x) = a\sin(b(x + c)) + d$.

3.5 Solution of trigonometric equations in a finite interval.
Equations of the type $a\sin(b(x + c)) = k$.
Equations leading to quadratic equations in, for example, $\sin x$.
Graphical interpretation of the above.

Introduction

The word *trigonometry* comes from two Greek words, *trigonon* and *metron*, meaning 'triangle measurement'. Trigonometry developed out of the use and study of triangles, in surveying, navigation, architecture and astronomy, to find relationships between lengths of sides of triangles and measurement of angles. As a result, trigonometric functions were initially defined as functions of angles – that is, functions with angle measurements as their domains. With the development of calculus in the seventeenth century and the growth of knowledge in the sciences, the application of trigonometric functions grew to include a wide variety of periodic (repetitive) phenomena such as wave motion, vibrating strings, oscillating pendulums, alternating electrical current and biological cycles. These applications of trigonometric functions require their domains to be sets of real numbers without reference to angles or triangles. Hence, trigonometry can be approached from two different perspectives – **functions of angles**, or **functions of real numbers.** The first perspective is the focus of the next chapter where trigonometric functions will be defined in terms of the **ratios of sides of a right triangle**. The second perspective is the focus of this chapter where trigonometric functions will be defined in terms of a real number that is the **length of an arc along the unit circle.** While it is possible to define trigonometric functions in these two different ways, they assign the same value (interpreted as an angle, an arc length, or simply a real number) to a particular real number. Although this chapter will not refer much to triangles, it seems fitting to begin by looking at angles and arc lengths – geometric objects indispensable to the two different ways of viewing trigonometry.

6.1 Angles, circles, arcs and sectors

Angles

An **angle** in a plane is made by rotating a ray about its endpoint, called the **vertex** of the angle. The starting position of the ray is called the **initial side** and the position of the ray after rotation is called the **terminal side** of the angle (Figure 6.1). An angle having its vertex at the origin and its initial side lying on the positive x-axis is said to be in **standard position** (Figure 6.2). A **positive angle** is produced when a ray is rotated in an anticlockwise direction, and a **negative angle** when a ray is rotated in a clockwise direction. Two angles in standard position that have the same terminal sides – regardless of the direction or number of rotations – are called **coterminal angles.** Greek letters are often used to represent angles, and the direction of rotation is indicated by an arc with an arrow at its endpoint. The x- and y-axes divide the coordinate plane into four quadrants (numbered with Roman numerals). Figure 6.3 shows a positive angle α (alpha) and a negative angle β (beta) that are coterminal in quadrant III.

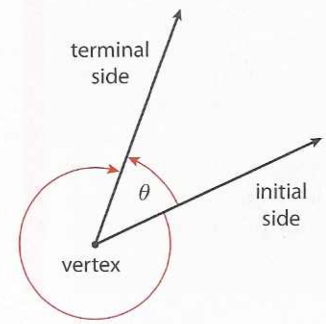


Figure 6.1

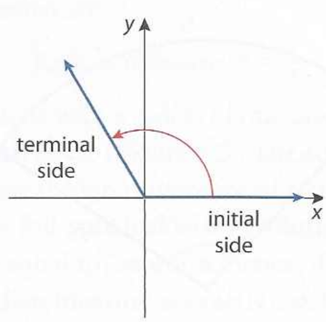


Figure 6.2 Standard position of an angle.

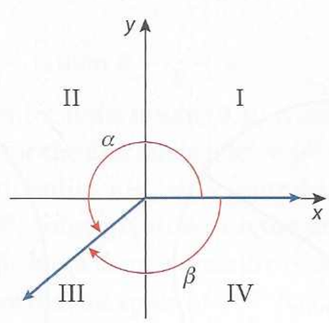


Figure 6.3 Coterminal angles.

Measuring angles: degree measure and radian measure

Perhaps the most natural unit for measuring large angles is the **revolution.** For example, most cars have an instrument (a tachometer) that indicates the number of revolutions per minute (rpm) at which the engine is operating. However, to measure smaller angles, we need a smaller unit. A common unit for measuring angles is the **degree**, of which there are 360 in one revolution. Hence, the unit of one degree (1°) is defined to be 1/360 of one anticlockwise revolution about the vertex.

The convention of having 360 degrees in one revolution can be traced back around 4000 years to ancient Babylonian civilizations. The number system most widely used today is a base 10, or **decimal**, system. Babylonian mathematics used a base 60, or **sexagesimal**, number system. Although 60 may seem to be an awkward number to have as a base, it does have certain advantages. It is the smallest number that has 2, 3, 4, 5 and 6 as factors – and it also has factors of 10, 12, 15, 20 and 30. But why 360 degrees? We're not certain but it may have to do with the Babylonians assigning 60 divisions to each angle in an equilateral triangle and exactly six equilateral triangles can be arranged around a single point. That makes $6 \times 60 = 360$ equal divisions in one full revolution. There are few numbers as small as 360 that have so many different factors. This makes the degree a useful unit for dividing one revolution into an equal number of parts. 120 degrees is $\frac{1}{3}$ of a revolution, 90 degrees is $\frac{1}{4}$ of a revolution, 60 degrees is $\frac{1}{6}$, 45 degrees is $\frac{1}{8}$, and so on.

There is another method of measuring angles that is more natural. Instead of dividing a full revolution into an arbitrary number of equal divisions (e.g. 360), consider an angle that has its vertex at the centre of a circle (a **central angle**) and subtends (or intercepts) a part of the circle, called an **arc of the circle.** Figure 6.4 shows three circles with radii of different lengths ($r_1 < r_2 < r_3$) and the same central angle θ (theta) subtending (intercepting) the arc lengths s_1, s_2 and s_3. Regardless of the size of the circle (i.e. length of the radius), the ratio of arc length (s) to radius (r) for a given circle will be constant. For the angle θ in Figure 6.4, $\frac{s_1}{r_1} = \frac{s_2}{r_2} = \frac{s_3}{r_3}$. Because this ratio is an arc length divided by another length (radius), it is just an ordinary real number and has no units.

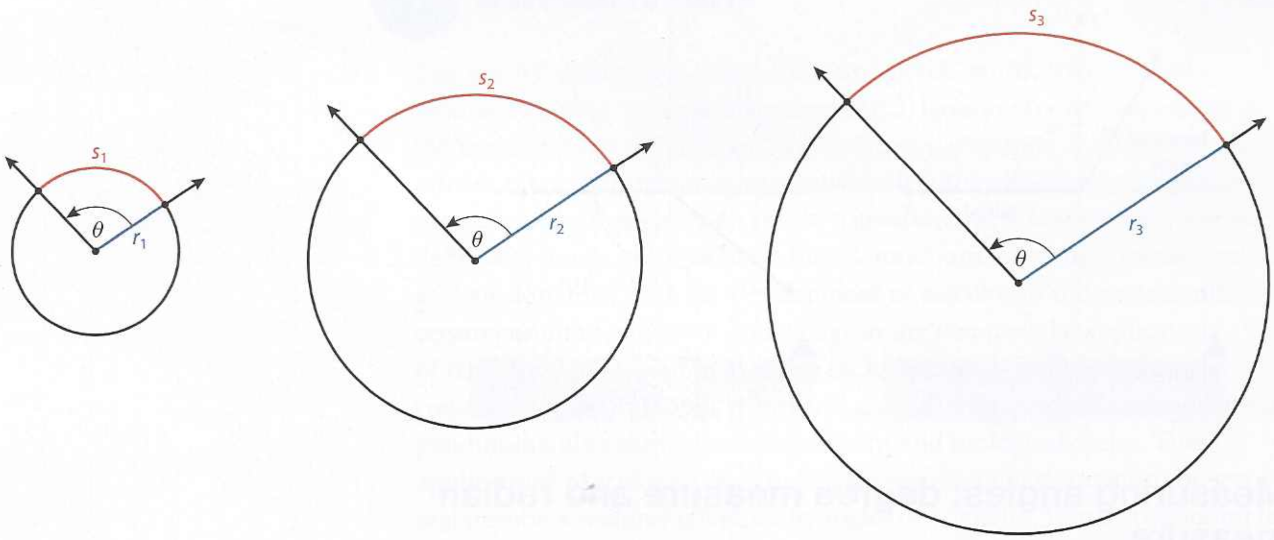


Figure 6.4 Different circles with the same central angle θ subtending different arcs, but the ratio of arc length to radius remains constant.

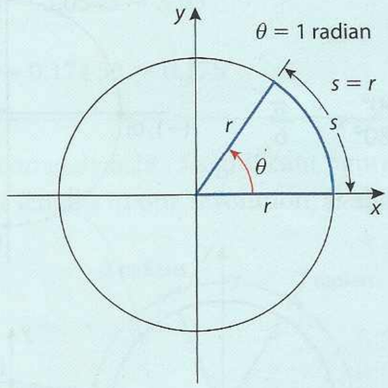
Minor and major arcs

If a central angle is **less** than 180°, the subtended arc is referred to as a **minor arc**. If a central angle is **greater** than 180°, the subtended arc is referred to as a **major arc**.

The ratio $\frac{s}{r}$ indicates how many radius lengths, r, fit into the length of the arc s. For example, if $\frac{s}{r} = 2$, the length of s is equal to two radius lengths. This accounts for the name **radian** and leads to the following definition.

Radian measure

One **radian** is the measure of a central angle θ of a circle that subtends an arc s of the circle that is exactly the same length as the radius r of the circle. That is, when $\theta = 1$ radian, arc length = radius.



When the measure of an angle is, for example, 5 radians, the word 'radians' does not indicate units (as when writing centimetres, seconds or degrees) but indicates the *method* of angle measurement. If the measure of an angle is in units of degrees, we must indicate this by word or symbol. For example, $\theta = 5$ degrees or $\theta = 5°$. However, when radian measure is used it is customary to write no units or symbol. For example, a central angle θ that subtends an arc equal to five radius lengths (radians) is simply given as $\theta = 5$.

The unit circle

When an angle is measured in radians it makes sense to draw it, or visualize it, so that it is in standard position. It follows that the angle will be a central angle of a circle whose centre is at the origin, as shown above. As Figure 6.4 illustrated, it makes no difference what size circle is used. The most practical circle to use is the circle with a radius of one unit so the radian measure of an angle will simply be equal to the length of the subtended arc.

Radian measure: $\theta = \frac{s}{r}$ If $r = 1$, then $\theta = \frac{s}{1} = s$.

The circle with a radius of one unit and centre at the origin (0, 0) is called the **unit circle** (Figure 6.5). The equation for the unit circle is $x^2 + y^2 = 1$. Because the circumference of a circle with radius r is $2\pi r$, a central angle of one full anticlockwise revolution (360°) subtends an arc on the unit circle equal to 2π units. Hence, if an angle has a degree measure of 360°, its radian measure is exactly 2π. It follows that an angle of 180° has a radian measure of exactly π. This fact can be used to convert between degree measure and radian measure, and vice versa.

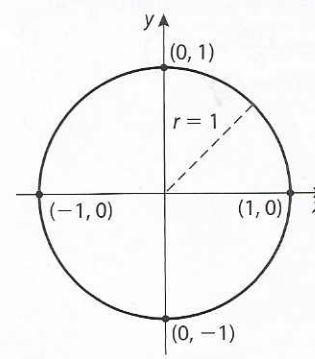


Figure 6.5 The unit circle.

Conversion between degrees and radians

Because $180° = \pi$ radians, $1° = \frac{\pi}{180}$ radians, and 1 radian $= \frac{180°}{\pi}$. An angle with a radian measure of 1 has a degree measure of approximately 57.3° (to 3 significant figures).

Example 1

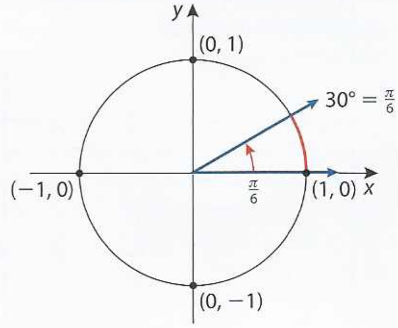
The angles of 30° and 45°, and their multiples, are often encountered in trigonometry. Convert 30° and 45° to radian measure and sketch the corresponding arc on the unit circle. Use these results to convert 60° and 90° to radian measure.

• **Hint:** It is very helpful to be able to quickly recall the results from Example 1:
$30° = \frac{\pi}{6}, 45° = \frac{\pi}{4}, 60° = \frac{\pi}{3}$
and $90° = \frac{\pi}{2}$. Of course, not all angles are multiples of 30° or 45° when expressed in degrees, and not all angles are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ when expressed in radians. However, these 'special' angles often appear in problems and applications. Knowing these four facts can help you to quickly convert mentally between degrees and radians for many common angles. For example, to convert 225° to radians, apply the fact that $225° = 5(45°)$. Since $45° = \frac{\pi}{4}$, then $225° = 5(45°) = 5\left(\frac{\pi}{4}\right) = \frac{5\pi}{4}$. And another example, convert $\frac{11\pi}{6}$ to degrees: $\frac{11\pi}{6} = 11\left(\frac{\pi}{6}\right) = 11(30°) = 330°$.

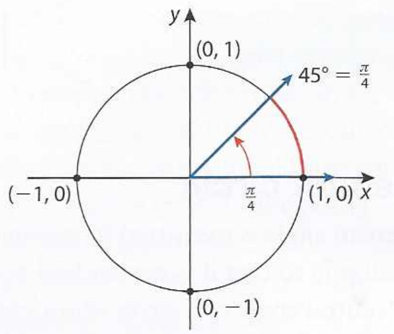
Solution

(Note that the 'degree' units cancel.)

$$30° = 30°\left(\frac{\pi}{180°}\right) = \frac{30°}{180°}\pi = \frac{\pi}{6}$$



$$45° = 45°\left(\frac{\pi}{180°}\right) = \frac{45°}{180°}\pi = \frac{\pi}{4}$$



Since $60° = 2(30°)$ and $30° = \frac{\pi}{6}$, then $60° = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$. Similarly, $90° = 2(45°)$ and $45° = \frac{\pi}{4}$, so $90° = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$.

Example 2

a) Convert the following radian measures to degrees. Express exactly, if possible. Otherwise, express accurate to 3 significant figures.
(i) $\frac{4\pi}{3}$ (ii) $-\frac{3\pi}{2}$ (iii) 5 (iv) 1.38

b) Convert the following degree measures to radians. Express exactly, if possible. Otherwise, express accurate to 3 significant figures.
(i) 135° (ii) −150° (iii) 175° (iv) 10°

Solution

a) (i) $\frac{4\pi}{3} = 4\left(\frac{\pi}{3}\right) = 4(60°) = 240°$

(ii) $-\frac{3\pi}{2} = -\frac{3}{2}(\pi) = -\frac{3}{2}(180°) = -270°$

(iii) $5\left(\frac{180°}{\pi}\right) \approx 286.479° \approx 286°$

(iv) $1.38\left(\frac{180°}{\pi}\right) \approx 79.068° \approx 79.1°$

• **Hint:** All GDCs will have a degree mode and a radian mode. Before doing any calculations with angles on your GDC, be certain that the mode setting for angle measurement is set correctly. Although you may be more familiar with degree measure, as you progress further in mathematics – and especially in calculus – radian measure is far more useful.

b) (i) $135° = 3(45°) = 3\left(\frac{\pi}{4}\right) = \frac{3\pi}{4}$

(ii) $-150° = -5(30°) = -5\left(\frac{\pi}{6}\right) = -\frac{5\pi}{6}$

(iii) $175°\left(\frac{\pi}{180°}\right) \approx 3.0543 \approx 3.05$

(iv) $10°\left(\frac{\pi}{180°}\right) \approx 0.174\,53 \approx 0.175$

Because 2π is approximately 6.28 (3 significant figures), there are a little more than six radius lengths in one revolution, as shown in Figure 6.6.

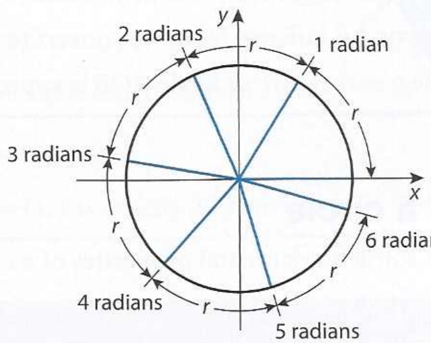


Figure 6.6

Arc length

For any angle θ, its radian measure is given by $\theta = \frac{s}{r}$. Simple rearrangement of this formula leads to another formula for computing arc length.

Arc length

For a circle of radius r, a central angle θ subtends an arc of the circle of length s given by

$$s = r\theta$$

where θ is in radian measure.

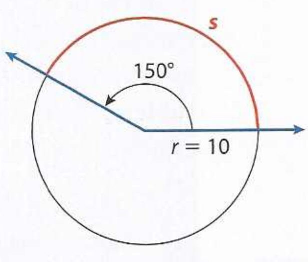
Example 3

A circle has a radius of 10 cm. Find the length of the arc of the circle subtended by a central angle of 150°.

Solution

To use the formula $s = r\theta$, we must first convert 150° to radian measure.

$$150° = 150°\left(\frac{\pi}{180°}\right) = \frac{150\pi}{180} = \frac{5\pi}{6}$$



Given that the radius, r, is 10 cm, substituting into the formula gives

$$s = r\theta \Rightarrow s = 10\left(\frac{5\pi}{6}\right) = \frac{25\pi}{3} \approx 26.179\,94$$

The length of the arc is approximately 26.18 cm (4 significant figures).

Note that the units of the product $r\theta$ are the same as the units of r because in radian measure θ has no units.

Example 4

The diagram shows a circle of centre O with radius $r = 6$ cm. Angle AOB subtends the minor arc AB such that the length of the arc is 10 cm. Find the measure of angle AOB in degrees to 3 significant figures.

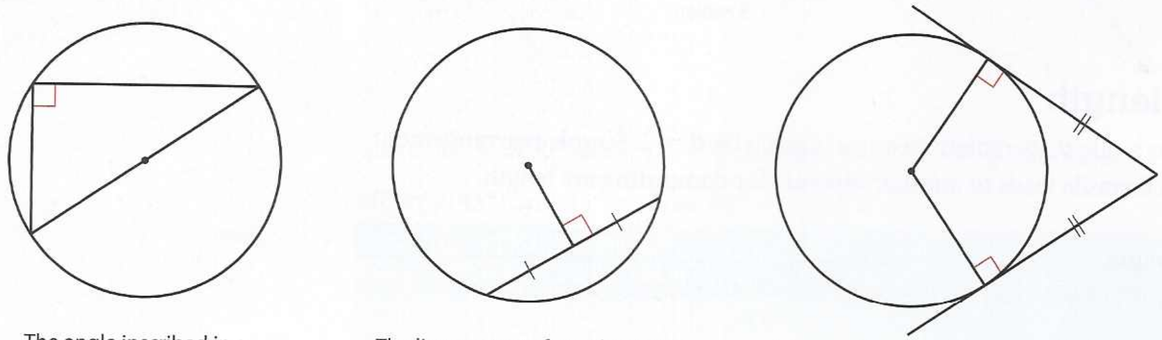
Solution

From the arc length formula, $s = r\theta$, we can state that $\theta = \frac{s}{r}$. Remember that the result for θ will be in radian measure. Therefore, angle $AOB = \frac{10}{6} = \frac{5}{3}$ or $1.\overline{6}$ radians. Now, we convert to degrees: $\frac{5}{3}\left(\frac{180^\circ}{\pi}\right)$ $\approx 95.492\,97^\circ$. The degree measure of angle AOB is approximately 95.5°.

Geometry of a circle

It is helpful to recall some fundamental properties of a circle (Figure 6.7).

Figure 6.7



The angle inscribed in a semicircle is a right angle.

The line segment from the centre perpendicular to a chord also bisects the chord.

A tangent to a circle is perpendicular to the radius drawn to the point of tangency.

If two tangents share an external point, the distances from the external point to the point of tangency are equal.

Sector of a circle

A **sector of a circle** is the region bounded by an arc of the circle and the two sides of a central angle (Figure 6.8). The ratio of the area of a sector to the area of the circle (πr^2) is equal to the ratio of the length of the subtended arc to the circumference of the circle ($2\pi r$). If s is the arc length and A is the area of the sector, we can write the following proportion: $\frac{A}{\pi r^2} = \frac{s}{2\pi r}$. Solving for A gives $A = \frac{\pi r^2 s}{2\pi r} = \frac{1}{2}rs$. From the formula for arc length we have $s = r\theta$, with θ the radian measure of the central angle. Substituting $r\theta$ for s gives the area of a sector to be $A = \frac{1}{2}rs = \frac{1}{2}r(r\theta) = \frac{1}{2}r^2\theta$. This result makes sense because, if the sector is the entire circle, $\theta = 2\pi$ and area $A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2(2\pi) = \pi r^2$, which is the formula for the area of a circle.

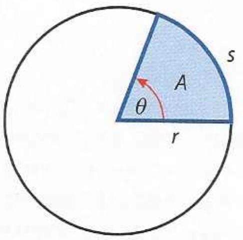


Figure 6.8 Sector of a circle.

Area of a sector

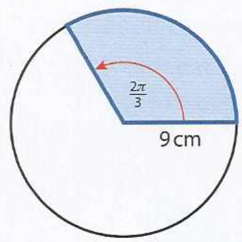
In a circle of radius r, the area of a sector with a central angle θ measured in radians is

$$A = \frac{1}{2}r^2\theta$$

Example 5

A circle of radius 9 cm has a sector whose central angle has radian measure $\frac{2\pi}{3}$. Find the exact values of the following: a) the length of the arc subtended by the central angle, and b) the area of the sector.

Solution



a) $s = r\theta \Rightarrow s = 9\left(\frac{2\pi}{3}\right) = 6\pi$

The length of the arc is exactly 6π cm.

b) $A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}(9)^2\left(\frac{2\pi}{3}\right) = 27\pi$

The area of the sector is exactly $27\pi\,\text{cm}^2$.

• **Hint:** The formula for arc length, $s = r\theta$, and the formula for area of a sector, $A = \frac{1}{2}r^2\theta$, are true only when θ is in radians.

Exercise 6.1

In questions 1–9, find the exact radian measure of the angle given in degree measure.

1 60° **2** 150° **3** −270°

4 36° **5** 135° **6** 50°

7 −45° **8** 400° **9** −480°

In questions 10–18, find the degree measure of the angle given in radian measure. If possible, express exactly. Otherwise, express accurate to 3 significant figures.

10 $\frac{3\pi}{4}$ **11** $-\frac{7\pi}{2}$ **12** 2

13 $\frac{7\pi}{6}$ **14** −2.5 **15** $\frac{5\pi}{3}$

16 $\frac{\pi}{12}$ **17** 1.57 **18** $\frac{8\pi}{3}$

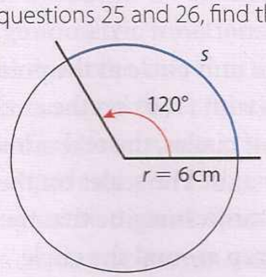
In questions 19–24, the measure of an angle in standard position is given. Find two angles – one positive and one negative – that are coterminal with the given angle. If no units are given, assume the angle is in radian measure.

19 30° **20** $\frac{3\pi}{2}$ **21** 175°

22 $-\frac{\pi}{6}$ **23** $\frac{5\pi}{3}$ **24** 3.25

In questions 25 and 26, find the length of the arc s in the figure.

25



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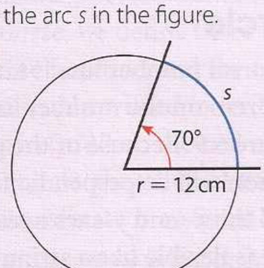
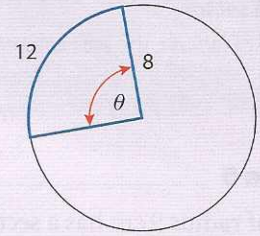
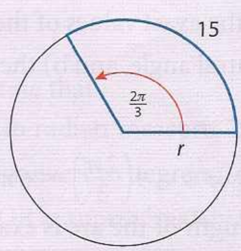


27 Find the angle θ in the figure in both radian measure and degree measure.

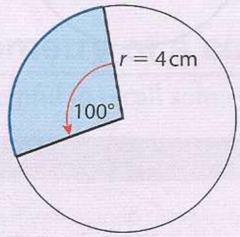


28 Find the radius r of the circle in the figure.



In questions 29 and 30, find the area of the sector in each figure.

29



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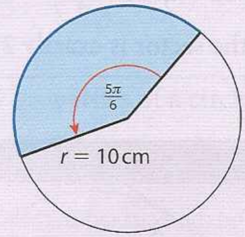


31 An arc of length 60 cm subtends a central angle α in a circle of radius 20 cm. Find the measure of α in both degrees and radians, approximate to 3 significant figures.

32 Find the length of an arc that subtends a central angle with radian measure of 2 in a circle of radius 16 cm.

33 The area of a sector of a circle with a central angle of 60° is 24 cm^2. Find the radius of the circle.

6.2 The unit circle and trigonometric functions

Several important functions can be described by mapping the coordinates of points on the real number line onto the points of the unit circle. Recall from the previous section that the unit circle has its centre at (0, 0), it has a radius of one unit and its equation is $x^2 + y^2 = 1$.

A wrapping function: the real number line and the unit circle

Suppose that the real number line is tangent to the unit circle at the point (1, 0) – and that zero on the number line matches with (1, 0) on the circle, as shown in Figure 6.9. Because of the properties of circles, the real number line in this position will be perpendicular to the x-axis. The scales on the number line and the x- and y-axes need to be the same. Imagine that the real number line is flexible like a string and can wrap around the circle, with zero on the number line remaining fixed to the point (1, 0) on the unit circle. When the top portion of the string moves along the circle, the wrapping is anticlockwise ($t > 0$), and when the bottom portion of the string moves along the circle, the wrapping is clockwise ($t < 0$). As the string wraps around the unit circle, each real number t on the string is mapped onto a point (x, y) on the circle. Hence, the real number line from 0 to t makes an arc of length t starting on the circle at $(1, 0)$ and ending at the point (x, y) on the circle. For example, since the circumference of the unit circle is 2π, the number $t = 2\pi$ will be wrapped anticlockwise around the circle to the point $(1, 0)$. Similarly, the number $t = \pi$ will be wrapped anticlockwise halfway around the circle to the point $(-1, 0)$ on the circle. And the number $t = -\frac{\pi}{2}$ will be wrapped clockwise one-quarter of the way around the circle to the point $(0, -1)$ on the circle. Note that each number t on the real number line is mapped (corresponds) to *exactly one* point on the unit circle, thereby satisfying the definition of a function (Section 2.1) – consequently this mapping is called a **wrapping function**.

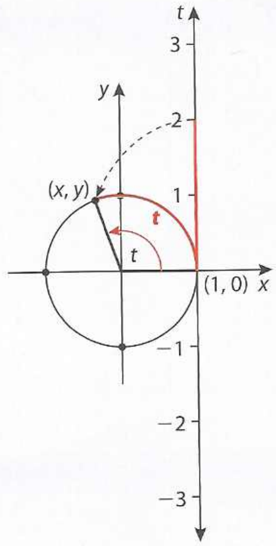


Figure 6.9

Before we leave our mental picture of the string (representing the real number line) wrapping around the unit circle, consider any pair of points on the string that are exactly 2π units from each other. Let these two points represent the real numbers t_1 and $t_1 + 2\pi$. Because the circumference of the unit circle is 2π, these two numbers will be mapped to the same point on the unit circle. Furthermore, consider the infinite number of points whose distance from t_1 is any integer multiple of 2π, i.e. $t_1 + k \cdot 2\pi, k \in \mathbb{Z}$, and again all of these numbers will be mapped to the same point on the unit circle. Consequently, the wrapping function is not a one-to-one function as defined in Section 2.3. Output for the function (points on the unit circle) are unchanged by the addition of any integer multiple of 2π to any input value (a real number). Functions that behave in such a repetitive (or cyclic) manner are called **periodic**.

> We are surrounded by periodic functions. A few examples include: the average daily temperature variation during the year; sunrise and the day of the year; animal populations over many years; the height of tides and the position of the Moon; and your height above ground when riding a Ferris wheel and the rotation of the wheel.

Definition of a periodic function

A function f such that $f(x) = f(x + p)$ is a **periodic function**. If p is the least positive constant for which $f(x) = f(x + p)$ is true, p is called the **period** of the function.

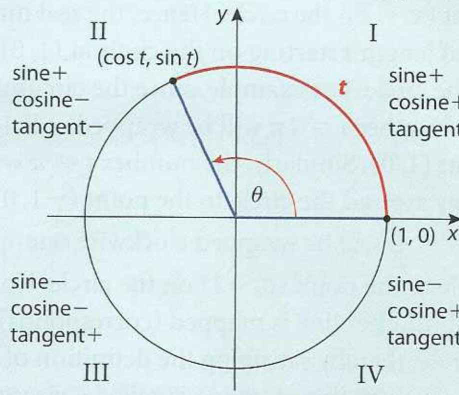
Trigonometric functions

From our discussions about functions in Chapter 2, it is customary for a function to have a domain (input) and range (output) that are sets having individual numbers as elements. We use the individual coordinates x and y of the points on the unit circle to define a certain set of functions called **trigonometric functions**. For this course, we define three trigonometric functions: the **sine** function, the **cosine** function and the **tangent** function. The names of these functions are often abbreviated in writing (but not speaking) as **sin**, **cos** and **tan**, respectively. When the real number t is wrapped to a point (x, y) on the unit circle, the value of the y-coordinate is assigned to the sine function; the x-coordinate is assigned to the cosine function; and the ratio of the two coordinates $\frac{y}{x}$ is assigned to the tangent function.

The trigonometric functions: sine, cosine and tangent

Let t be any real number and (x, y) a point on the unit circle to which t is mapped. Then the function definitions are:

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{\sin t}{\cos t} = \frac{y}{x}, x \neq 0$$



On the unit circle: $x = \cos t$, $y = \sin t$.

Signs of the trigonometric functions depend on the quadrant where the arc t terminates.

• **Hint:** When sine, cosine and tangent are defined as circular functions based on the unit circle, radian measure is used. The values for the domain of the sine and cosine functions are real numbers that are arc lengths on the unit circle. As we know from the previous section, the arc length on the unit circle subtends an angle in standard position, whose radian measure is equivalent to the arc length (see definition box above).

Because the definitions for the sine, cosine and tangent functions given here do not refer to triangles or angles, but rather to a real number representing an arc length on the unit circle, the name **circular functions** is also given to them. In fact, from this chapter's perspective that these functions are *functions of real numbers* rather than *functions of angles*, 'circular' is a more appropriate adjective than 'trigonometric'. Nevertheless, trigonometric is the more common label and will be used throughout the book.

Let's use the definitions for these three trigonometric, or circular, functions to evaluate them for some 'easy' values of t.

Example 6

Evaluate the sine, cosine and tangent functions for the following values of t.

a) $t = 0$ b) $t = \frac{\pi}{2}$ c) $t = \pi$

d) $t = \frac{3\pi}{2}$ e) $t = 2\pi$

Solution

Evaluating the sin, cos and tan functions for any value of t involves finding the coordinates of the point on the unit circle where the arc of length t will 'wrap to' (or terminate), starting at the point $(1, 0)$. It is useful to remember that an arc of length π is equal to one-half of the circumference of the unit circle. All of the values for t in this example are positive, so the arc length will wrap along the unit circle in an anticlockwise direction.

a) An arc of length $t = 0$ has no length so it 'terminates' at the point $(1, 0)$. Therefore, by definition

$$\sin 0 = y = 0$$

$$\cos 0 = x = 1$$

$$\tan 0 = \frac{y}{x} = \frac{0}{1} = 0$$

b) An arc of length $t = \frac{\pi}{2}$ is equivalent to one-quarter of the circumference of the unit circle (Figure 6.10), so it terminates at the point (0, 1). By definition:

$$\sin\frac{\pi}{2} = y = 1$$

$$\cos\frac{\pi}{2} = x = 0$$

$$\tan\frac{\pi}{2} = \frac{y}{x} = \frac{1}{0} \text{ which is undefined}$$

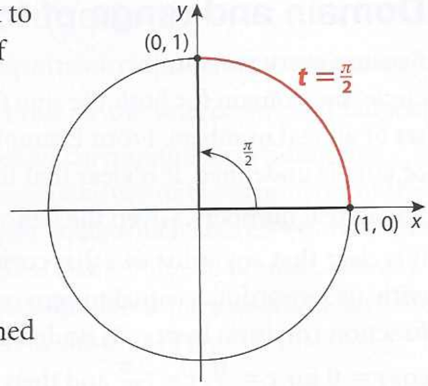


Figure 6.10

c) An arc of length $t = \pi$ is equivalent to one-half of the circumference of the unit circle (Figure 6.11), so it terminates at the point $(-1, 0)$. By definition:

$$\sin \pi = y = 0$$

$$\cos \pi = x = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$$

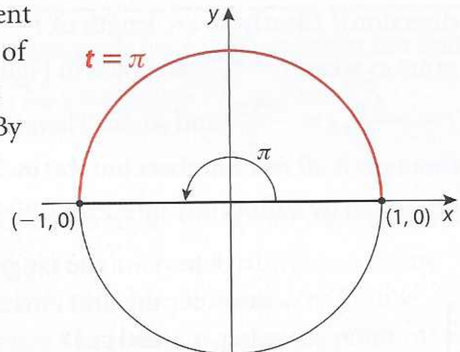


Figure 6.11

d) An arc of length $t = \frac{3\pi}{2}$ is equivalent to three-quarters of the circumference of the unit circle (Figure 6.12), so it terminates at the point $(0, -1)$.
By definition:

$$\sin\frac{3\pi}{2} = y = -1$$

$$\cos\frac{3\pi}{2} = x = 0$$

$$\tan\frac{3\pi}{2} = \frac{y}{x} = \frac{-1}{0} \text{ which is undefined}$$

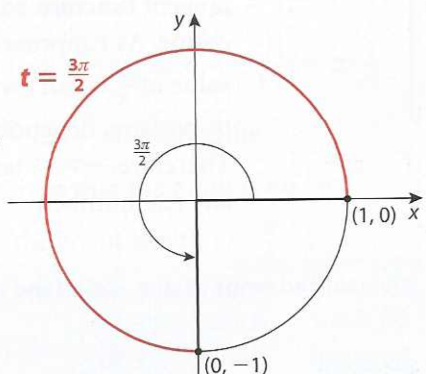


Figure 6.12

e) An arc of length $t = 2\pi$ is equivalent to the circumference of the unit circle (Figure 6.13), so it terminates at the point $(1, 0)$. By definition:

$$\sin 2\pi = y = 0$$

$$\cos 2\pi = x = 1$$

$$\tan 2\pi = \frac{y}{x} = \frac{0}{1} = 0$$

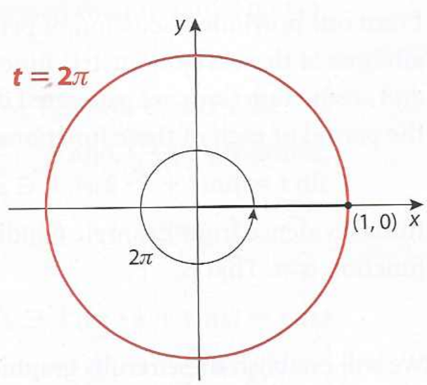


Figure 6.13

Domain and range of trigonometric functions

Because every real number t corresponds to exactly one point on the unit circle, the domain for both the sine function and the cosine function is the set of all real numbers. From Example 6, parts b) and d), where the value of $\tan t$ is undefined, it is clear that the domain for the tangent function is not all real numbers. Given the definitions $\tan t = \frac{y}{x}, x \neq 0$, and $\cos t = x$, it is clear that any value of t that corresponds to a point on the unit circle with an x-coordinate equal to zero cannot be in the domain of the tangent function (division by zero is undefined). From Example 6, we can see that $\cos t = 0$ for $t = \frac{\pi}{2}$, $t = \frac{3\pi}{2}$ and then for $t = \frac{5\pi}{2}$, and for $t = \frac{7\pi}{2}$, and so on. What about negative values for t (arc lengths wrapped in a clockwise direction)? Clearly an arc length of $t = -\frac{\pi}{2}$ will terminate at $(0, -1)$, the same as when $t = \frac{3\pi}{2}$, as shown in Figure 6.14. And $\cos t = 0$ also for $t = -\frac{3\pi}{2}$, $t = -\frac{5\pi}{2}$, and so on. Therefore, the domain of the tangent function is all real numbers but *not* including the infinite set of numbers generated by adding any integer multiple of π to $\frac{\pi}{2}$.

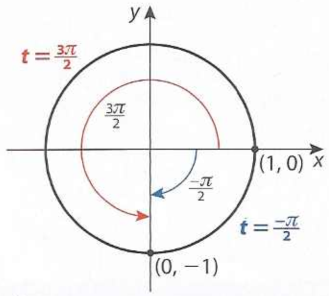


Figure 6.14

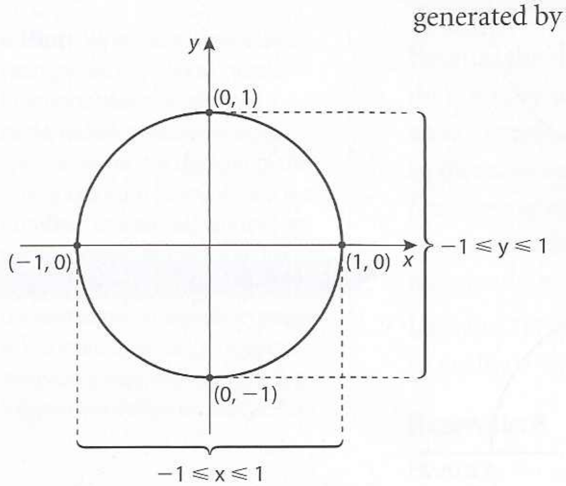


Figure 6.15

To determine the range of the sine and cosine functions, consider the unit circle shown in Figure 6.15. Because $\sin t = y$ and $\cos t = x$ and (x, y) is on the unit circle, we can see that $-1 \leq y \leq 1$ and $-1 \leq x \leq 1$. Therefore, $-1 \leq \sin t \leq 1$ and $-1 \leq \cos t \leq 1$. The range for the tangent function will not be bounded as for sine and cosine. As t approaches values where $x = \cos t = 0$, the value of $\frac{y}{x} = \tan t$ will become very large – either negative or positive, depending on which quadrant t is in. Therefore, $-\infty < \tan t < \infty$; or, in other words, $\tan t$ can be any real number.

Domain and range of sine, cosine and tangent functions

$f(t) = \sin t$	domain: $\{t : t \in \mathbb{R}\}$	range: $-1 \leq f(t) \leq 1$
$f(t) = \cos t$	domain: $\{t : t \in \mathbb{R}\}$	range: $-1 \leq f(t) \leq 1$
$f(t) = \tan t$	domain: $\left\{t : t \in \mathbb{R}, t \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$	range: $f(t) \in \mathbb{R}$

From our previous discussion of periodic functions, we can conclude that all three of these trigonometric functions are periodic. Given that the sine and cosine functions are generated directly from the wrapping function, the period of each of these functions is 2π. That is,

$$\sin t = \sin(t + k \cdot 2\pi),\ k \in \mathbb{Z} \text{ and } \cos t = \cos(t + k \cdot 2\pi),\ k \in \mathbb{Z}$$

Initial evidence from Example 6 indicates that the period of the tangent function is π. That is,

$$\tan t = \tan(t + k \cdot \pi),\ k \in \mathbb{Z}$$

We will establish these results graphically in the next section. Also note that since these functions are periodic then they are not one-to-one functions.

Evaluating trigonometric functions

In Example 6, the unit circle was divided into four equal arcs corresponding to t values of $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π. Let's evaluate the sine, cosine and tangent functions for further values of t that would correspond to dividing the unit circle into eight equal arcs. Let's also make use of the symmetry of the unit circle. That is, any points on the unit circle which are reflections about the x-axis will have the same x-coordinate (same value of cosine), and any points on the unit circle which are reflections about the y-axis will have the same y-coordinate (same value of sine), as shown in Figure 6.16.

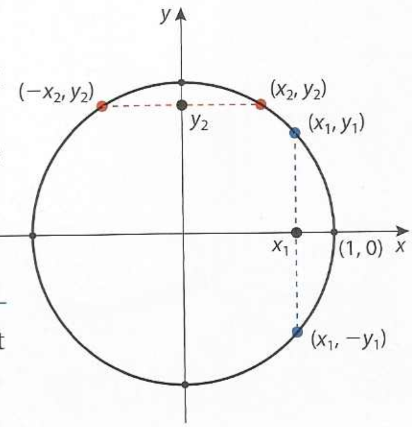


Figure 6.16

Example 7

Evaluate the sine, cosine and tangent functions for $t = \frac{\pi}{4}$, and then use that result to evaluate the same functions for $t = \frac{3\pi}{4}$, $t = \frac{5\pi}{4}$ and $t = \frac{7\pi}{4}$.

Solution

When an arc of length $t = \frac{\pi}{4}$ is wrapped along the unit circle starting at $(1, 0)$, it will terminate at a point (x_1, y_1) in quadrant I that is equidistant from $(1, 0)$ and $(0, 1)$. Since the line $y = x$ is a line of symmetry for the unit circle, (x_1, y_1) is on this line. Hence, the point (x_1, y_1) is the point of intersection of the unit circle $x^2 + y^2 = 1$ with the line $y = x$. Let's find the coordinates of the intersection point by solving this pair of simultaneous equations by substituting x for y into the equation $x^2 + y^2 = 1$.

$$x^2 + y^2 = 1 \Rightarrow x^2 + x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \tfrac{1}{2} \Rightarrow x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$$

Rationalising the denominator gives $x = \pm\frac{\sqrt{2}}{2}$ and, since the point is in the first quadrant, $x = \frac{\sqrt{2}}{2}$. Given that the point is on the line $y = x$ then $y = \frac{\sqrt{2}}{2}$. Therefore, the arc of length $t = \frac{\pi}{4}$ will terminate at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ on the unit circle. Using the symmetry of the unit circle, we can also determine the points on the unit circle where arcs of length $t = \frac{3\pi}{4}$, $t = \frac{5\pi}{4}$ and $t = \frac{7\pi}{4}$ terminate. These arcs and the coordinates of their terminal points are given in Figure 6.17.

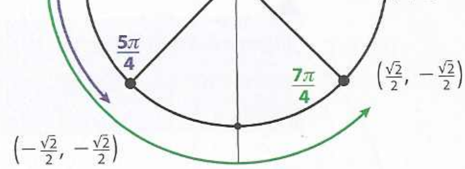


Figure 6.17

Using the coordinates of these points, we can now evaluate the trigonometric functions for $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$. By definition:

$$t = \frac{\pi}{4}: \quad \sin\frac{\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \cos\frac{\pi}{4} = x = \frac{\sqrt{2}}{2} \qquad \tan\frac{\pi}{4} = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$t = \frac{3\pi}{4}: \quad \sin\frac{3\pi}{4} = y = \frac{\sqrt{2}}{2} \qquad \cos\frac{3\pi}{4} = x = -\frac{\sqrt{2}}{2} \qquad \tan\frac{3\pi}{4} = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = -1$$

$$t = \frac{5\pi}{4}\text{: } \sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2} \quad \cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2} \quad \tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$

$$t = \frac{7\pi}{4}\text{: } \sin\frac{7\pi}{4} = y = -\frac{\sqrt{2}}{2} \quad \cos\frac{7\pi}{4} = x = \frac{\sqrt{2}}{2} \quad \tan\frac{7\pi}{4} = \frac{y}{x} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$$

We can use a method similar to that of Example 7 to find the point on the unit circle where an arc of length $t = \frac{\pi}{6}$ terminates in the first quadrant. Then we can again apply symmetry about the line $y = x$ and the y- and x-axes to find points on the circle corresponding to arcs whose lengths are multiples of $\frac{\pi}{6}$, e.g. $\frac{2\pi}{6} = \frac{\pi}{3}, \frac{4\pi}{6} = \frac{2\pi}{3}$, etc. Arcs whose lengths are multiples of $\frac{\pi}{4}$ and $\frac{\pi}{6}$ correspond to eight equally spaced points and twelve equally spaced points, respectively, around the unit circle, as shown in Figures 6.18 and 6.19. The coordinates of these points give us the sine, cosine and tangent values for common values of t.

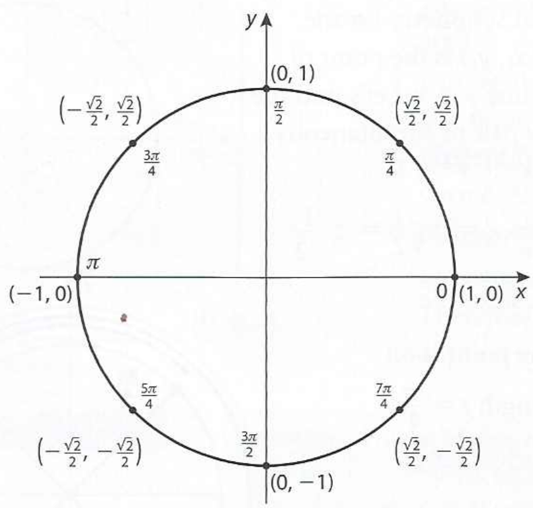


Figure 6.18 Arc lengths that are multiples of $\frac{\pi}{4}$ divide the unit circle into eight equally spaced points.

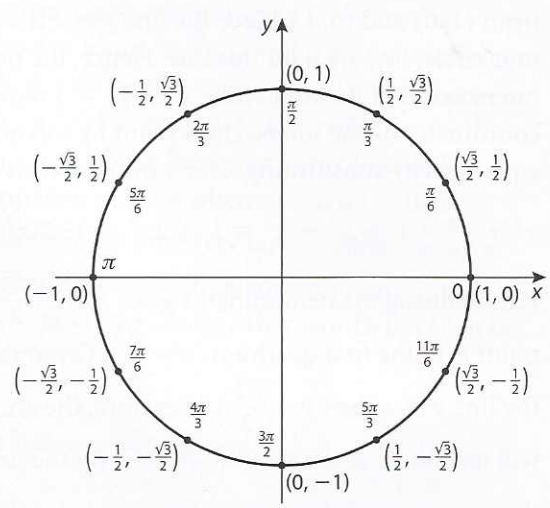


Figure 6.19 Arc lengths that are multiples of $\frac{\pi}{6}$ divide the unit circle into twelve equally spaced points.

You will find it very helpful to know from memory the exact values of sine and cosine for numbers that are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Use the unit circle diagrams shown in Figures 6.18 and 6.19 as a guide to help you do this and to visualize the location of the terminal points of different arc lengths. With the symmetry of the unit circle and a point's location in the coordinate plane telling us the sign of x and y (see definition box page 170), we only need to remember the sine and cosine of common values of t in the first quadrant and on the positive x- and y-axes. These are organized in Table 6.1.

t	$\sin t$	$\cos t$	$\tan t$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

Table 6.1 The sine, cosine and tangent of common values of t.

If t is not a multiple of one of these common values, the values of the trigonometric functions for that number can be found using your GDC.

Example 8

Find the following function values. Find the exact value, if possible. Otherwise, find the approximate value accurate to 3 significant figures.

a) $\sin\frac{2\pi}{3}$ b) $\cos\frac{5\pi}{4}$ c) $\tan\frac{11\pi}{6}$ d) $\sin\frac{13\pi}{6}$ e) $\cos 3.75$

Solution

a) The terminal point for $\frac{2\pi}{3}$ is in the second quadrant and is the reflection in the y-axis of the terminal point for $\frac{\pi}{3}$, whose y-coordinate is $\frac{\sqrt{3}}{2}$. Therefore, $\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$.

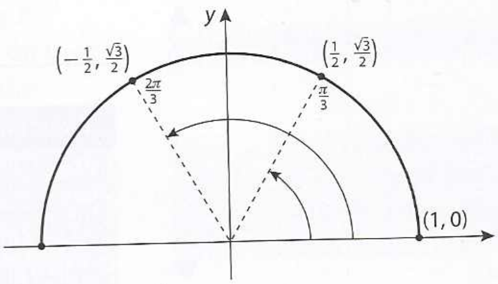


Figure 6.20

b) $\frac{5\pi}{4}$ is in the third quadrant. Hence, its x-coordinate and cosine must be negative. All of the odd multiples of $\frac{\pi}{4}$ have terminal points with x- and y-coordinates of $\pm\frac{\sqrt{2}}{2}$. Therefore, $\cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$.

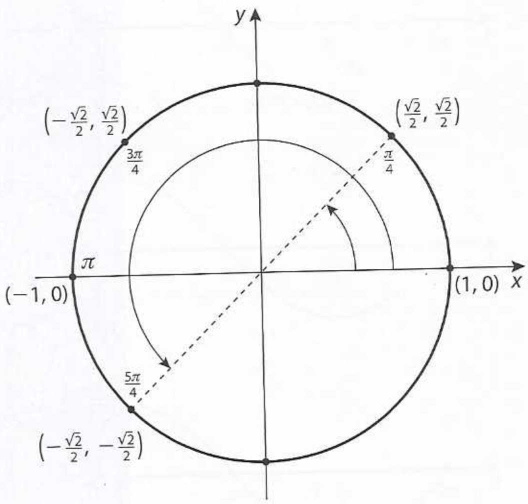


Figure 6.21

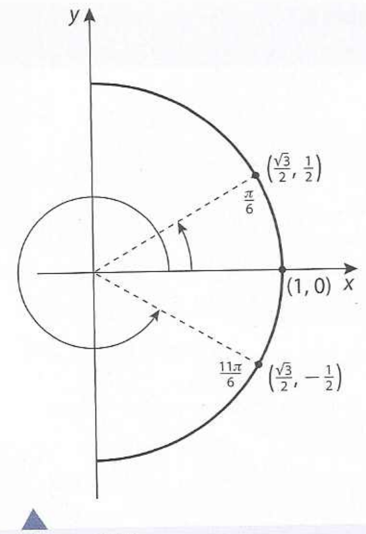


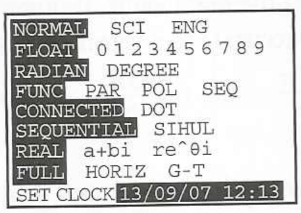
Figure 6.22

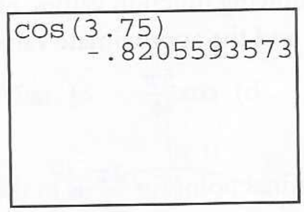
c) $\frac{11\pi}{6}$ is in the fourth quadrant, so its tangent will be negative. Its terminal point is the reflection in the x-axis of the terminal point for $\frac{\pi}{6}$, whose coordinates are $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Therefore,

$$\tan\frac{11\pi}{6} = \frac{y}{x} = \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$

d) $\frac{13\pi}{6}$ is more than one revolution. Because $\frac{13\pi}{6} = \frac{\pi}{6} + 2\pi$ and the period of the sine function is 2π [i.e. $\sin t = \sin(t + k\cdot 2\pi)$, $k \in \mathbb{Z}$] then $\sin\frac{13\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$.

e) An arc of length 3.75 will have its terminal point in the third quadrant since $\pi \approx 3.14$ and $\frac{3\pi}{2} \approx 4.71$, meaning $\pi < 3.75 < \frac{3\pi}{2}$. Hence, $\cos 3.75$ must be negative. To evaluate $\cos 3.75$ you must use your GDC. Be certain that it is set to radian mode. To an accuracy of 3 significant figures, $\cos 3.75 \approx -0.821$.





Exercise 6.2

In questions 1–9, t is the length of an arc on the unit circle starting from (1, 0)
a) State the quadrant in which the terminal point of the arc lies. b) Find the coordinates of the terminal point (x, y) on the unit circle. Give exact values for x and y, if possible. Otherwise, approximate values to 3 significant figures.

1 $t = \frac{\pi}{6}$ **2** $t = \frac{5\pi}{3}$ **3** $t = \frac{7\pi}{4}$

4 $t = \frac{3\pi}{2}$ **5** $t = 2$ **6** $t = -\frac{\pi}{4}$

7 $t = -1$ **8** $t = -\frac{5\pi}{4}$ **9** $t = 3.52$

In questions 10–18, state the exact value (if possible) of the sine, cosine and tangent of the given real number.

10 $\frac{\pi}{3}$ **11** $\frac{5\pi}{6}$ **12** $-\frac{3\pi}{4}$

13 $\frac{\pi}{2}$ **14** $-\frac{4\pi}{3}$ **15** 3π

16 $\frac{3\pi}{2}$ **17** $-\frac{7\pi}{6}$ **18** $t = 1.25\pi$

In questions 19–22, use the periodic properties of the sine and cosine functions to find the exact value of $\sin x$ and $\cos x$.

19 $x = \frac{13\pi}{6}$ **20** $x = \frac{10\pi}{3}$

21 $x = \frac{15\pi}{4}$ **22** $x = \frac{17\pi}{6}$

6.3 Graphs of trigonometric functions

The graph of a function provides a useful visual image of its behaviour. For example, from the previous section we know that trigonometric functions are periodic, i.e. their values repeat in a regular manner. The graphs of the trigonometric functions should provide a picture of this periodic behaviour. In this section, we will graph the sine, cosine and tangent functions and transformations of the sine and cosine functions.

```
sin(2.53)
          .5741721484
sin(2.53+2π)
          .5741721484
sin(2.53+4π)
          .5741721484
```

The period of $y = \sin x$ is 2π.

Graphs of the sine and cosine functions

Since the period of the sine function is 2π, we know that two values of t (domain) that differ by 2π (e.g. $\frac{\pi}{6}$ and $\frac{13\pi}{6}$ in Example 8) will produce the same value for y (range). This means that any portion of the graph of $y = \sin t$ with a t-interval of length 2π (called one **period** or **cycle** of the graph) will repeat. Remember that the domain of the sine function is all real numbers, so one period of the graph of $y = \sin t$ will repeat indefinitely in the positive and negative direction. Therefore, in order to construct a complete graph of $y = \sin t$, we need to graph just one period of the function, that is, from $t = 0$ to $t = 2\pi$, and then repeat the pattern in both directions.

We know from the previous section that $\sin t$ is the y-coordinate of the terminal point on the unit circle corresponding to the real number t (Figure 6.23). In order to generate one period of the graph of $y = \sin t$, we need to record the y-coordinates of a point on the unit circle and the corresponding value of t as the point travels anticlockwise one revolution, starting from the point $(1, 0)$. These values are then plotted on a graph with t on the horizontal axis and y (i.e. $\sin t$) on the vertical axis. Figure 6.24 illustrates this process in a sequence of diagrams.

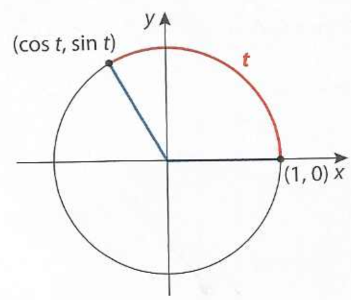
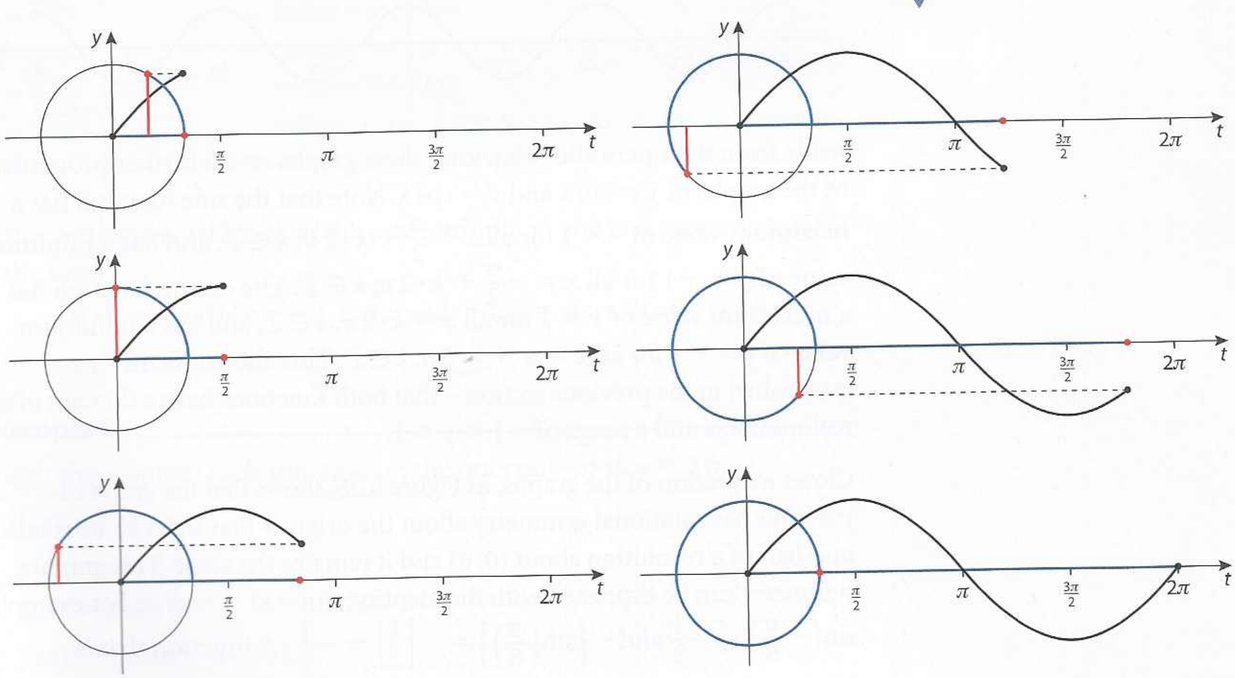


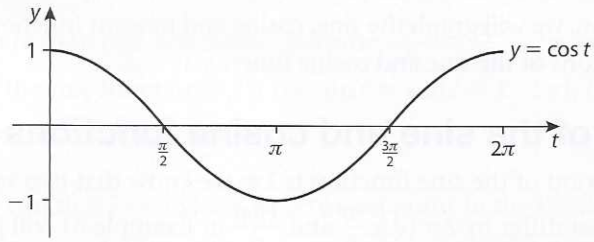
Figure 6.23

Figure 6.24 Graph of the sine function for $0 \leq t \leq 2\pi$ generated from a point travelling along the unit circle.



As the point ($\cos t$, $\sin t$) travels along the unit circle, the x-coordinate (i.e. $\cos t$) goes through the same cycle of values as the y-coordinate ($\sin t$) does. The only difference is that the x-coordinate begins at a different value in the cycle – when $t = 0$, $y = 0$, but $x = 1$. The result is that the graph of $y = \cos t$ is the exact same shape as $y = \sin t$ but it has been shifted to the left $\frac{\pi}{2}$ units. The graph of $y = \cos t$ for $0 \leqslant t \leqslant 2\pi$ is shown in Figure 6.25.

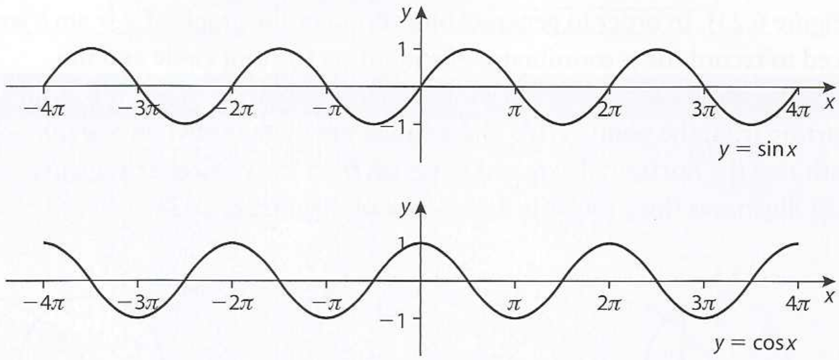
Figure 6.25



The convention is to use the letter x to denote the variable in the domain of the function. Hence, we will use the letter x rather than t and write the trigonometric functions as $y = \sin x$, $y = \cos x$ and $y = \tan x$.

Because the period for both the sine function and cosine function is 2π, to graph $y = \sin x$ and $y = \cos x$ for wider intervals of x we simply need to repeat the shape of the graph that we generated from the unit circle for $0 \leqslant x \leqslant 2\pi$ (Figures 6.24 and 6.25). Figure 6.26 shows the graphs of $y = \sin x$ and $y = \cos x$ for $-4\pi \leqslant x \leqslant 4\pi$.

Figure 6.26



Aside from their periodic behaviour, these graphs reveal further properties of the graphs of $y = \sin x$ and $y = \cos x$. Note that the sine function has a maximum value of $y = 1$ for all $x = \frac{\pi}{2} + k \cdot 2\pi$, $k \in \mathbb{Z}$, and has a minimum value of $y = -1$ for all $x = -\frac{\pi}{2} + k \cdot 2\pi$, $k \in \mathbb{Z}$. The cosine function has a maximum value of $y = 1$ for all $x = k \cdot 2\pi$, $k \in \mathbb{Z}$, and has a minimum value of $y = -1$ for all $x = \pi + k \cdot 2\pi$, $k \in \mathbb{Z}$. This also confirms – as established in the previous section – that both functions have a domain of all real numbers and a range of $-1 \leqslant y \leqslant 1$.

Closer inspection of the graphs, in Figure 6.26, shows that the graph of $y = \sin x$ has rotational symmetry about the origin – that is, it can be rotated one-half of a revolution about (0, 0) and it remains the same. This graph symmetry can be expressed with the identity: $\sin(-x) = -\sin x$. For example, $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ and $-\left[\sin\left(\frac{\pi}{6}\right)\right] = -\left[\frac{1}{2}\right] = -\frac{1}{2}$. A function that is

symmetric about the origin is called an **odd function**. The graph of $y = \cos x$ has line symmetry in the y-axis – that is, it can be reflected in the line $x = 0$ and it remains the same. This graph symmetry can be expressed with the identity: $\cos(-x) = \cos x$. For example, $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$.
A function that is symmetric about the y-axis is called an **even function.**

Odd and even functions

A function is **odd** if, for each x in the domain of f, $f(-x) = -f(x)$.

The graph of an odd function is symmetric with respect to the origin (rotational symmetry).

A function is **even** if, for each x in the domain of f, $f(-x) = f(x)$.

The graph of an even function is symmetric with respect to the y-axis (line symmetry).

Graphs of transformations of the sine and cosine functions

In Section 2.4, we learned how to transform the graph of a function by horizontal and vertical translations, by reflections in the coordinate axes, and by stretching and shrinking – both horizontal and vertical. The following is a review of these transformations.

Review of transformations of graphs of functions

Assume that a, b, c and d are real numbers.

To obtain the graph of:	**From the graph of $y = f(x)$:**
$y = f(x) + d$	Translate d units up for $d > 0$, d units down for $d < 0$.
$y = f(x + c)$	Translate c units left for $c > 0$, c units right for $c < 0$.
$y = -f(x)$	Reflect in the x-axis.
$y = af(x)$	Vertical stretch ($a > 1$) or shrink ($0 < a < 1$) of factor a.
$y = f(-x)$	Reflect in the y-axis.
$y = f(bx)$	Horizontal stretch ($0 < b < 1$) or shrink ($b > 1$) of factor $\frac{1}{b}$.

In this section, we will look at the composition of sine and cosine functions of the form

$$f(x) = a\sin[b(x + c)] + d \quad \text{and} \quad f(x) = a\cos[b(x + c)] + d$$

Example 9

Sketch the graph of each function on the interval $-\pi \leqslant x \leqslant 3\pi$.

a) $f(x) = 2\cos x$

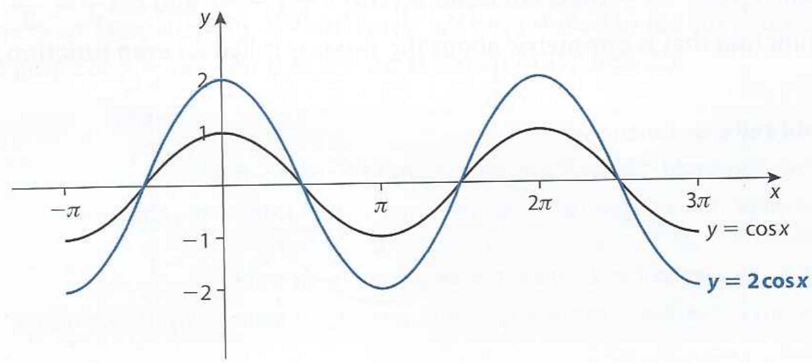
b) $g(x) = \cos x + 3$

c) $h(x) = 2\cos x + 3$

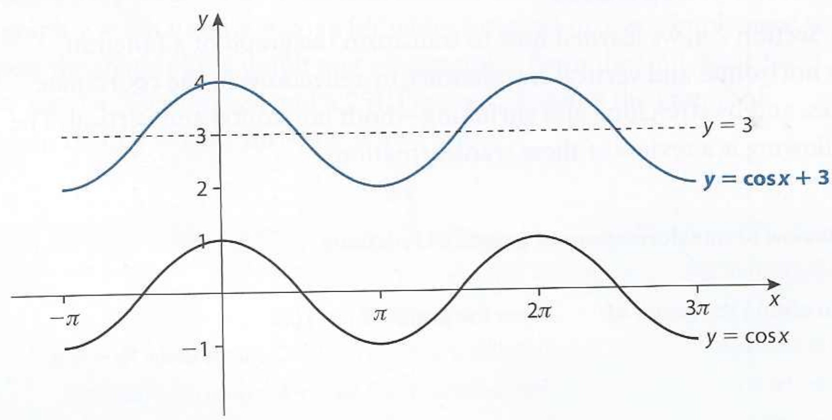
d) $p(x) = \frac{1}{2}\sin x - 2$

Solution

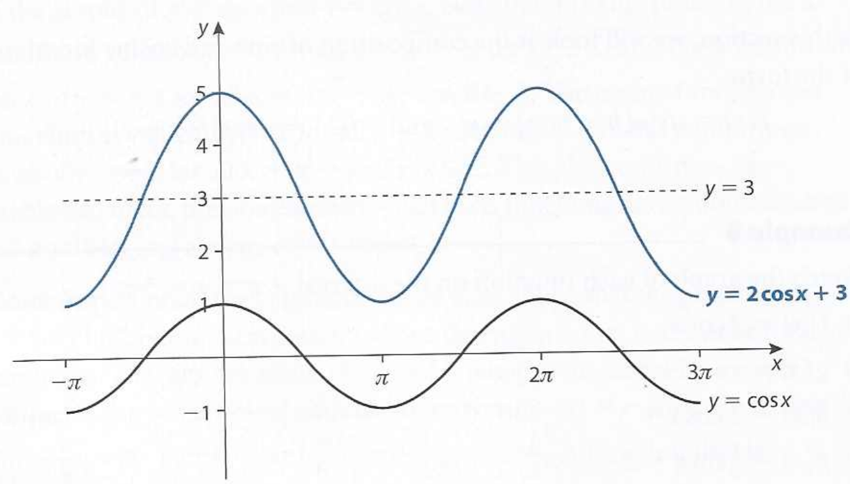
a) Since $a = 2$, the graph of $y = 2\cos x$ is obtained by vertically stretching the graph of $y = \cos x$ by a factor of 2.



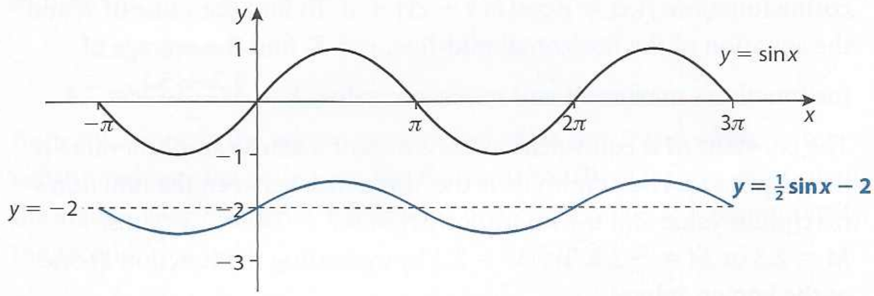
b) Since $d = 3$, the graph of $y = \cos x + 3$ is obtained by translating the graph of $y = \cos x$ three units up.



c) We can obtain the graph of $y = 2\cos x + 3$ by combining both of the transformations to the graph of $y = \cos x$ performed in parts a) and b) – namely, a vertical stretch of factor 2 and a translation 3 units up.



d) The graph of $y = \frac{1}{2}\sin x - 2$ can be obtained by vertically shrinking the graph of $y = \sin x$ by a factor of $\frac{1}{2}$ and then translating it down 2 units.



In part a), the graph of $y = 2\cos x$ has many of the same properties as the graph of $y = \cos x$: same period, and the maximum and minimum values occur at the same values of x. However, the graph ranges between -2 and 2 instead of -1 and 1. This difference is best described by referring to the **amplitude** of each graph. The amplitude of $y = \cos x$ is 1 and the amplitude of $y = 2\cos x$ is 2. The amplitude of a sine or cosine graph is not always equal to its maximum value. In part b), the amplitude of $y = \cos x + 3$ is 1; in part c), the amplitude of $y = 2\cos x + 3$ is 2; and the amplitude of $y = \frac{1}{2}\sin x - 2$ is $\frac{1}{2}$. For all three of these, the graphs oscillate about the horizontal line $y = d$. How *high* and *low* the graph oscillates with respect to the mid-line, $y = d$, is the graph's amplitude. With respect to the general form $y = af(x)$, changing the amplitude is equivalent to a vertical stretching or shrinking. Thus, we can give a more precise definition of amplitude in terms of the parameter a.

Amplitude of the graph of sine and cosine functions

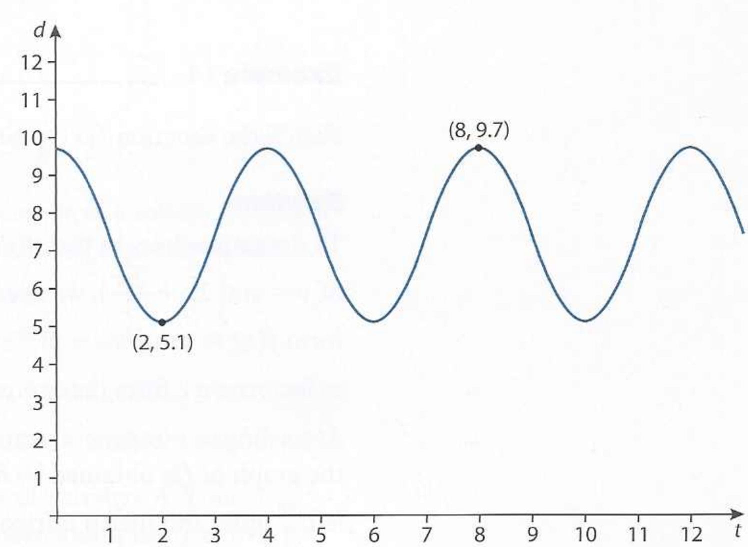
The graphs of $f(x) = a\sin[b(x + c)] + d$ and $f(x) = a\cos[b(x + c)] + d$ have an **amplitude** equal to $|a|$.

Example 10

Waves are produced in a long tank of water. The depth of the water, d metres, at t seconds, at a fixed location in the tank, is modelled by the function $d(t) = M\cos\left(\frac{\pi}{2}t\right) + K$, where M and K are positive constants. On the right is the graph of $d(t)$ for $0 \leq t \leq 12$ indicating that the point $(2, 5.1)$ is a minimum and the point $(8, 9.7)$ is a maximum.

a) Find the value of K and the value of M.

b) After $t = 0$, find the first time when the depth of the water is 9.7 metres.



Solution

a) The constant K is equivalent to the constant d in the general form of a cosine function: $f(x) = a\cos[b(x + c)] + d$. To find the value of K and the equation of the horizontal mid-line, $y = K$, find the average of the function's maximum and minimum value: $K = \frac{9.7 + 5.1}{2} = 7.4$.

The constant M is equivalent to the constant a whose absolute value is the amplitude. The amplitude is the difference between the function's maximum value and the mid-line: $|M| = 9.7 - 7.4 = 2.3$. Thus, $M = 2.3$ or $M = -2.3$. Try $M = 2.3$ by evaluating the function at one of the known values:

$d(2) = 2.3\cos\left(\frac{\pi}{2}(2)\right) + 7.4 = 2.3\cos\pi + 7.4 = 2.3(-1) + 7.4 = 5.1.$

This agrees with the point $(2, 5.1)$ on the graph. Therefore, $M = 2.3$.

b) Maximum values of the function $(d(8) = 9.7)$ occur at values of t that differ by a value equal to the period. From the graph, we can see that the difference in t values from the minimum $(2, 5.1)$ to the maximum $(8, 9.7)$ is equivalent to one-and-a-half periods. Therefore, the period is 4 and the first time after $t = 0$ at which $d = 9.7$ is $t = 4$.

All four of the functions in Example 9 had the same period of 2π, but the function in Example 10 had a period of 4. Because $y = \sin x$ completes one period from $x = 0$ to $x = 2\pi$, it follows that $y = \sin bx$ completes one period from $bx = 0$ to $bx = 2\pi$. This implies that $y = \sin bx$ completes one period from $x = 0$ to $x = \frac{2\pi}{b}$. This agrees with the period for the function $d(t) = 2.3\cos\left(\frac{\pi}{2}t\right) + 7.4$ in Example 10: period $= \frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{2}} = \frac{2\pi}{1}\cdot\frac{2}{\pi} = 4.$

Note that the change in amplitude and vertical translation had no effect on the period. We should also expect that a horizontal translation of a sine or cosine curve should not affect the period. The next example looks at a function that is horizontally translated (shifted) and has a period different from 2π.

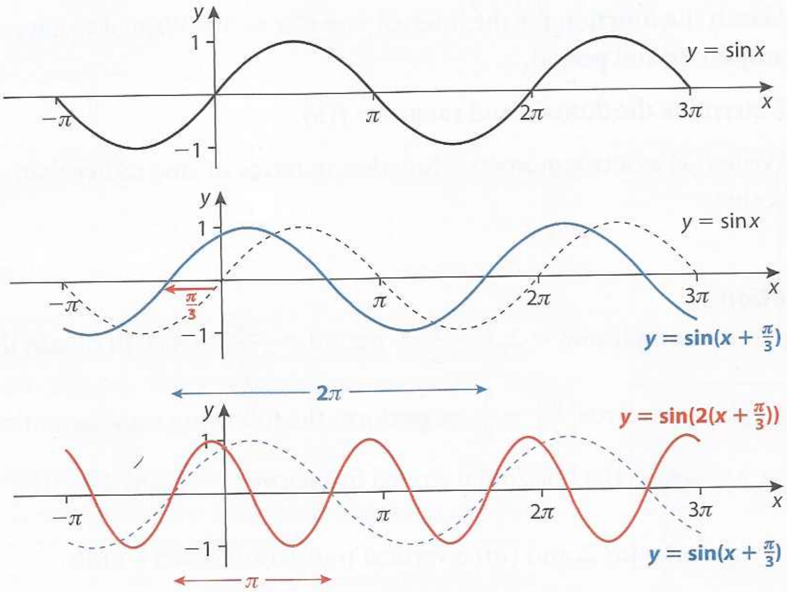
Example 11

Sketch the function $f(x) = \sin\left(2x + \frac{2\pi}{3}\right)$.

Solution

To determine how to transform the graph of $y = \sin x$ to obtain the graph of $y = \sin\left(2x + \frac{2\pi}{3}\right)$, we need to make sure the function is written in the form $f(x) = a\sin[b(x + c)] + d$. Clearly, $a = 1$ and $d = 0$, but we will need to factorize a 2 from the expression $2x + \frac{2\pi}{3}$ to get $f(x) = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$. According to our general transformations from Chapter 2, we expect that the graph of f is obtained by first translating the graph of $y = \sin x$ to the left $\frac{\pi}{3}$ units and then a horizontal shrinking by factor $\frac{1}{2}$ (see Section 2.4).

The graphs below illustrate the two-stage sequence of transforming $y = \sin x$ to $y = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$.



Note: A horizontal translation of a sine or cosine curve is often referred to as a **phase shift**. The equations $y = \sin\left(x + \frac{\pi}{3}\right)$ and $y = \sin\left[2\left(x + \frac{\pi}{3}\right)\right]$ both underwent a phase shift of $-\frac{\pi}{3}$.

Period and horizontal translation (phase shift) of sine and cosine functions

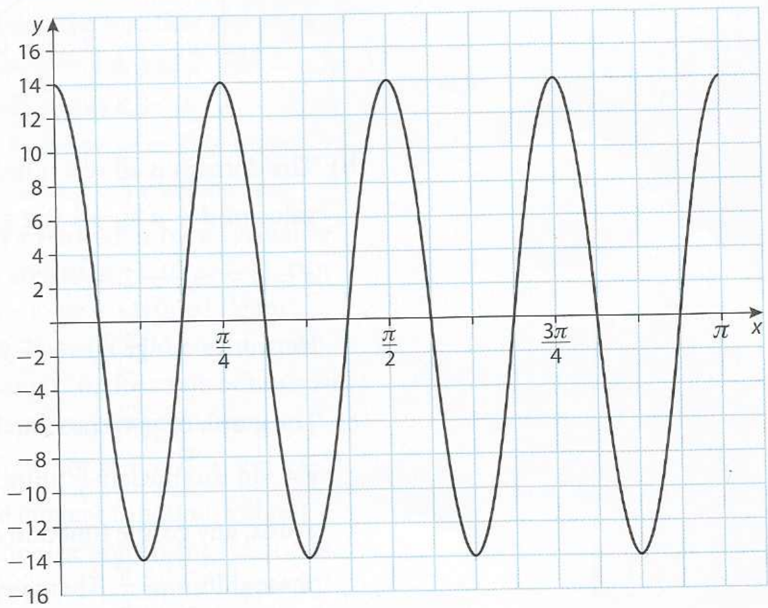
Given that b is a positive real number, $y = a\sin[b(x + c)] + d$ and $y = a\cos[b(x + c)] + d$ have a **period** of $\frac{2\pi}{b}$ and a horizontal translation (**phase shift**) of $-c$.

Example 12

The graph of a function in the form $y = a\cos bx$ is given in the diagram right.

a) Write down the value of a.

b) Calculate the value of b.



Solution

a) The amplitude of the graph is 14. Therefore, $a = 14$.

b) From inspecting the graph we can see that the period is $\frac{\pi}{4}$.

$$\text{Period} = \frac{2\pi}{b} = \frac{\pi}{4}$$

$$b\pi = 8\pi \Rightarrow b = 8.$$

Example 13

For the function $f(x) = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2}$:

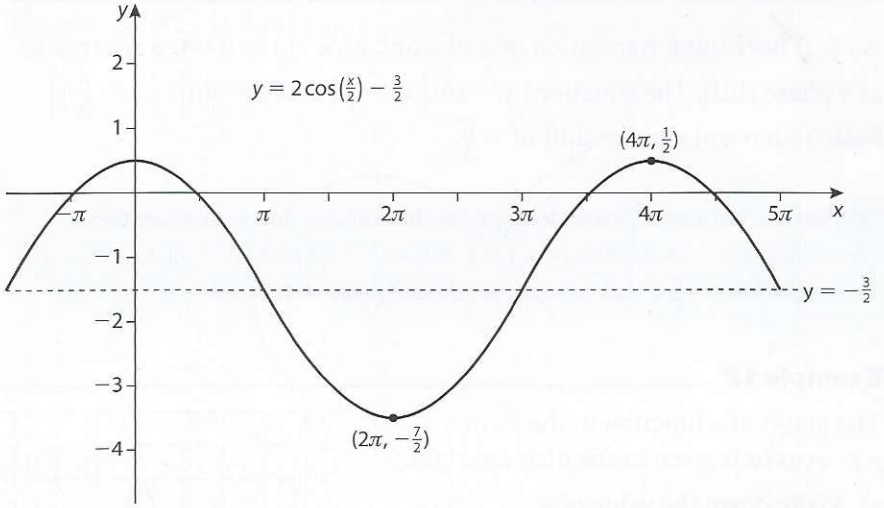
a) Sketch the function for the interval $-\pi \leqslant x \leqslant 5\pi$. Write down its amplitude and period.

b) Determine the domain and range for $f(x)$.

c) Write $f(x)$ as a trigonometric function in terms of sine rather than cosine.

Solution

a) $a = 2 \Rightarrow$ amplitude $= 2$; $b = \frac{1}{2} \Rightarrow$ period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$. To obtain the graph of $y = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2}$, we perform the following transformations on $y = \cos x$: (i) a horizontal stretch by factor $\frac{1}{\frac{1}{2}} = 2$, (ii) a vertical stretch by factor 2, and (iii) a vertical translation down $\frac{3}{2}$ units.



b) The domain is all real numbers. The function will reach a maximum value of $d + a = -\frac{3}{2} + 2 = \frac{1}{2}$, and a minimum value of $d - a = -\frac{3}{2} - 2 = -\frac{7}{2}$.

Hence, the range is $-\frac{7}{2} \leqslant y \leqslant \frac{1}{2}$.

c) The graph of $y = \cos x$ can be obtained by translating the graph of $y = \sin x$ to the left $\frac{\pi}{2}$ units. Thus, $\cos x = \sin\left(x + \frac{\pi}{2}\right)$, or, in other words, any cosine function can be written as a sine function with a phase shift $= -\frac{\pi}{2}$. Therefore, $f(x) = 2\cos\left(\frac{x}{2}\right) - \frac{3}{2} = 2\sin\left(\frac{x}{2} + \frac{\pi}{2}\right) - \frac{3}{2}$.

Horizontal translation (phase shift) identities

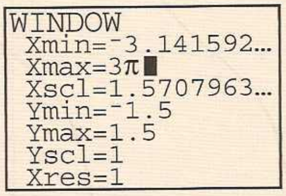
The following are true for all values of x:

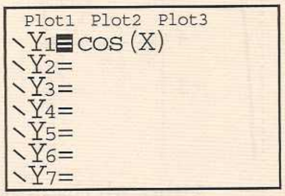
$\cos x = \sin\left(x + \frac{\pi}{2}\right)$ $\qquad$ $\sin x = \cos\left(x - \frac{\pi}{2}\right)$

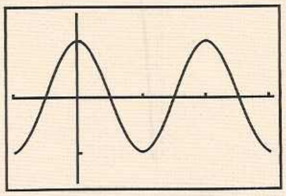
$\cos x = \sin\left(\frac{\pi}{2} - x\right)$ $\qquad$ $\sin x = \cos\left(\frac{\pi}{2} - x\right)$

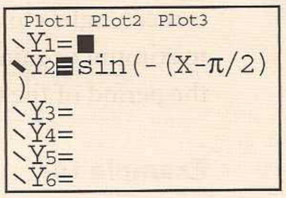
The identity $\cos x = \sin\left(x + \frac{\pi}{2}\right)$ is equivalent to the identity $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ because $\sin\left(\frac{\pi}{2} - x\right) = \sin\left[-\left(x - \frac{\pi}{2}\right)\right]$ and the graph of $y = \sin\left[-\left(x - \frac{\pi}{2}\right)\right]$ can be obtained by first translating $y = \sin x$ to the right $\frac{\pi}{2}$ units, and then reflecting the graph in the y-axis. This produces the same graph as $y = \cos x$. This can be confirmed nicely on your GDC as shown. Therefore, $\cos x = \sin\left(\frac{\pi}{2} - x\right)$. In fact, it is also true that $\sin x = \cos\left(\frac{\pi}{2} - x\right)$. Clearly, $x + \left(\frac{\pi}{2} - x\right) = \frac{\pi}{2}$. If the domain ($x$) values were being treated as angles, then x and $\frac{\pi}{2} - x$ would be complementary angles.

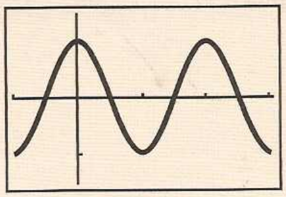
This is why cosine is considered the co-function of sine. Two trigonometric functions f and g are co-functions if the following are true for all x: $f(x) = g\left(\frac{\pi}{2} - x\right)$ and $f\left(\frac{\pi}{2} - x\right) = g(x)$.











Graph of the tangent function

From work done earlier in this chapter, we expect that the behaviour of the tangent function will be significantly different from that of the sine and cosine functions. In Section 6.2, we concluded that the function $f(x) = \tan x$ has a domain of all real numbers such that $x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, and that its range is all real numbers. Also, the results for Example 6 in Section 6.2 led us to speculate that the period of the tangent function is π. This makes sense since the identity $\tan x = \frac{\sin x}{\cos x}$ informs us that $\tan x$ will be zero whenever $\sin x = 0$, which occurs at values of x that differ by π (visualize arcs on the unit circle whose terminal points are either $(1, 0)$ or $(-1, 0)$). The values of x for which $\cos x = 0$ cause $\tan x$ to be undefined ('gaps' in the domain) also differ by π (the points $(0, 1)$ or $(0, -1)$ on the unit circle). As x approaches these values where $\cos x = 0$, the value of $\tan x$ will become very large – either very large negative or very large positive.

Thus, the graph of $y = \tan x$ has vertical asymptotes at $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

Consequently, the graphical behaviour of the tangent function will not be a wave pattern such as that produced by the sine and cosine functions, but rather a series of separate curves that repeat every π units. Figure 6.27 shows the graph of $y = \tan x$ for $-2\pi \leq x \leq 2\pi$.

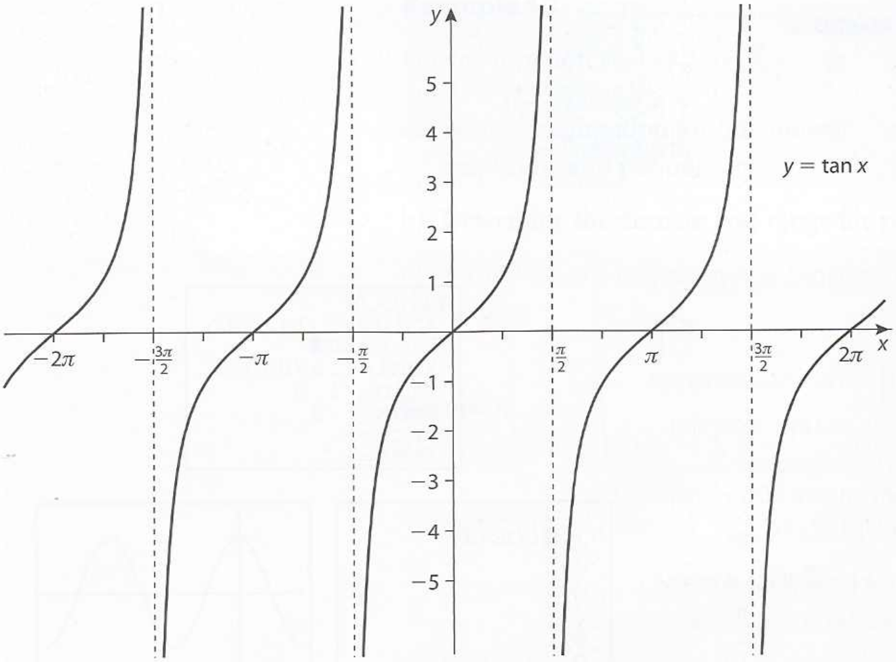


Figure 6.27

The graph gives clear confirmation that the period of the tangent function is π, that is, $\tan x = \tan(x + k \cdot \pi)$, $k \in \mathbb{Z}$.

The graph of $y = \tan x$ has rotational symmetry about the origin – that is, it can be rotated one-half of a revolution about (0, 0) and it remains the same. Hence, like the sine function, tangent is an odd function and $\tan(-x) = -\tan x$.

Although the graph of $y = \tan x$ can undergo a vertical stretch or shrink, it is meaningless to consider its amplitude since the tangent function has no maximum or minimum value. However, other transformations can affect the period of the tangent function.

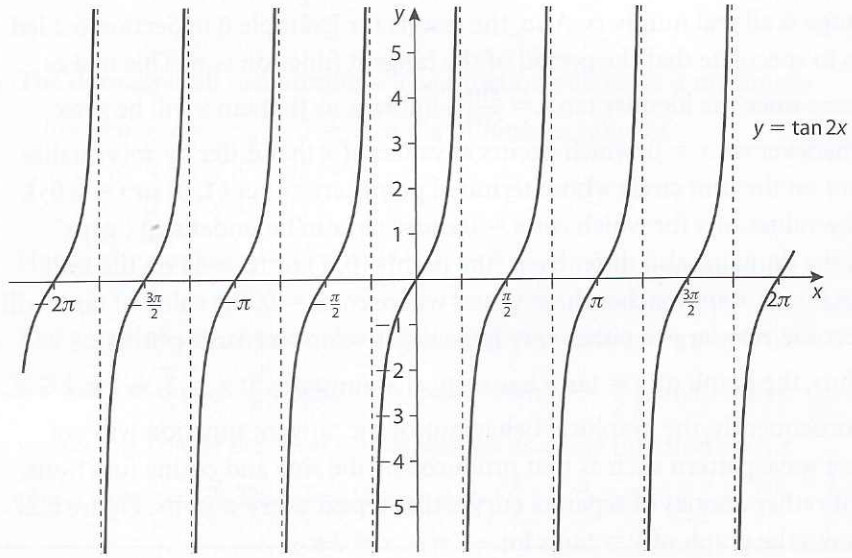
Example 14

Sketch each function.

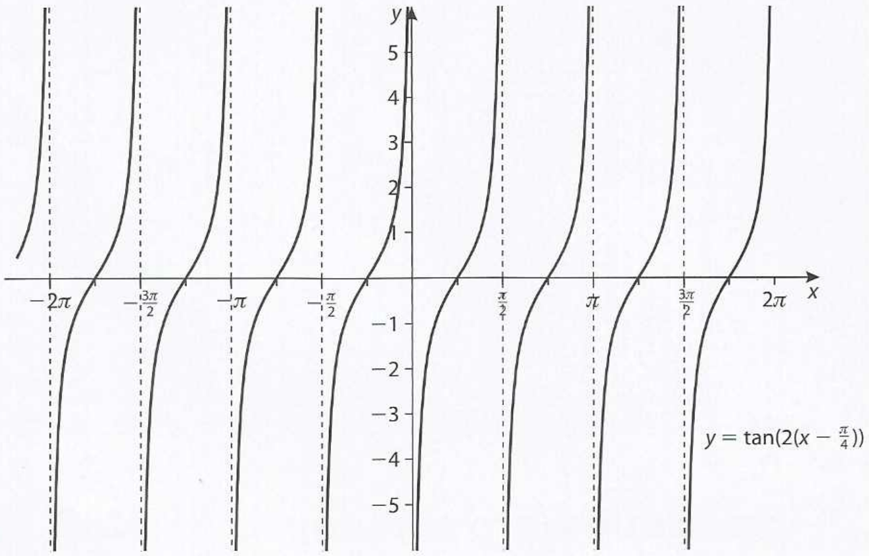
a) $f(x) = \tan 2x$ b) $g(x) = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$

Solution

a) An equation in the form $y = f(bx)$ indicates a horizontal shrinking of $f(x)$ by a factor of $\frac{1}{b}$. Hence, the period of $y = \tan 2x$ is $\frac{1}{2} \cdot \pi = \frac{\pi}{2}$.



b) The graph of $y = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$ is obtained by first translating the graph of $y = \tan x$ to the right $\frac{\pi}{4}$ units, and then a horizontal shrinking by a factor of $\frac{1}{2}$. As for $f(x) = \tan 2x$ in part a), the period of $g(x) = \tan\left[2\left(x - \frac{\pi}{4}\right)\right]$ is $\frac{\pi}{2}$.



Exercise 6.3

In questions 1–9, without using your GDC, sketch a graph of each equation on the interval $-\pi \leqslant x \leqslant 3\pi$.

1 $y = 2\sin x$

2 $y = \cos x - 2$

3 $y = \frac{1}{2}\cos x$

4 $y = \sin\left(x - \frac{\pi}{2}\right)$

5 $y = \cos(2x)$

6 $y = 1 + \tan x$

7 $y = \sin\left(\frac{x}{2}\right)$

8 $y = \tan\left(x + \frac{\pi}{2}\right)$

9 $y = \cos\left(2x - \frac{\pi}{4}\right)$

For each function in questions 10–12:

a) Sketch the function for the interval $-\pi \leqslant x \leqslant 5\pi$. Write down its amplitude and period.

b) Determine the domain and range for $f(x)$.

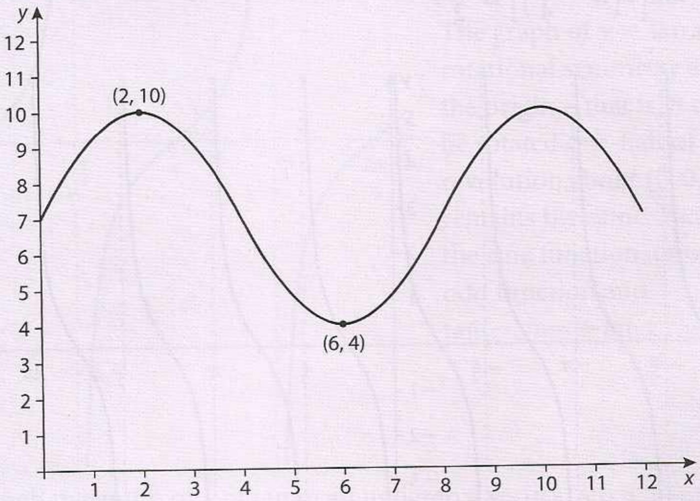
10 $f(x) = \frac{1}{2}\cos x - 3$

11 $g(x) = 3\sin(3x) - \frac{1}{2}$

12 $g(x) = 1.2\sin\left(\frac{x}{2}\right) + 4.3$

In questions 13–15, a graph for the interval $0 \leqslant x \leqslant 12$ is given for a trigonometric equation that can be written in the form $y = A\sin\left(\frac{\pi}{4}x\right) + B$. Two points – one a minimum and the other a maximum – are indicated on the graph. Find the value of A and B for each.

13



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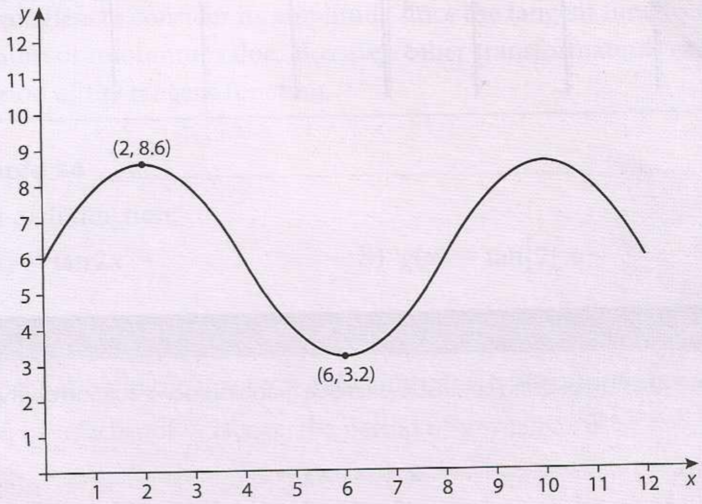

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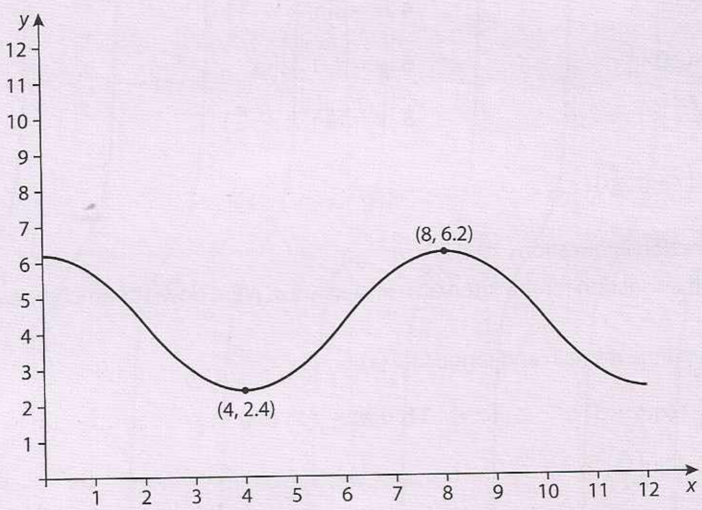
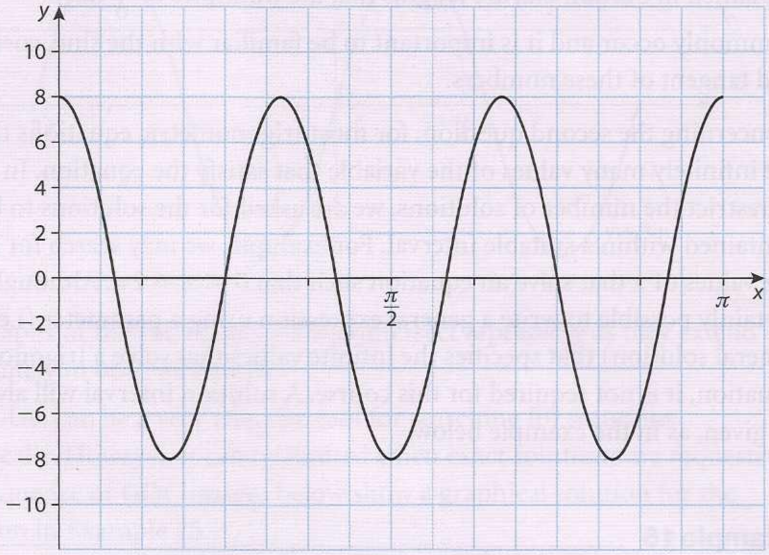


16 The graph of a function in the form $y = p\cos qx$ is given in the diagram below.

a) Write down the value of p. b) Calculate the value of q.



6.4 Solving trigonometric equations and trigonometric identities

The primary focus of this section is to examine methods for solving equations that contain the sine, cosine and tangent functions. For example, the following are **trigonometric equations**:

$\sin x = \frac{1}{2}$ $3\cos x = 5\sin x$ $\tan x = \frac{\sin x}{\cos x}$ $1 + \sin x = 3\cos^2 x$ $\sin^2 x + \cos^2 x = 1$

The equations $\tan x = \frac{\sin x}{\cos x}$ and $\sin^2 x + \cos^2 x = 1$ are examples of special equations called **identities**. As we learned in Section 1.6, an identity is an equation that is true for all possible values of the variable. The other equations are true for only certain values of x. Identities can be helpful in solving trigonometric equations by allowing us to simplify some trigonometric expressions. Equations that contain trigonometric functions often can be solved using the same graphical and algebraic methods that solve other equations.

The unit circle and exact solutions to trigonometric equations

When you are asked to solve a trigonometric equation, there are two important questions you need to consider:

1. Is it possible, or required, to express any solution(s) exactly?
2. For what interval of the variable (usually x) are all solutions to be found?

With regard to the first question, exact solutions are only attainable, in most cases, if they are an integer multiple of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. The variable for which we are trying to solve in trigonometric equations is a real number that can be interpreted as the length of an arc on the unit circle. As explained in Section 6.2, arc lengths that are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$ commonly occur and it is important to be familiar with the sine, cosine and tangent of these numbers.

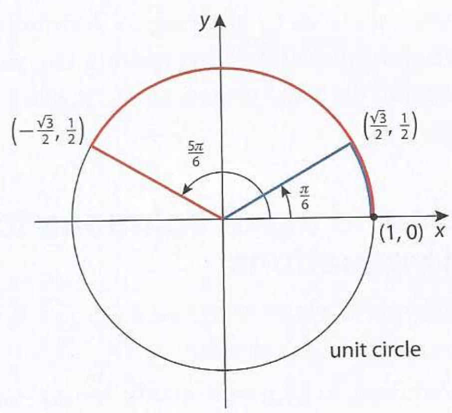
Concerning the second question, for most trigonometric equations there are infinitely many values of the variable that satisfy the equation. In order to restrict the number of solutions, we are asked for the solutions to be contained within a suitable interval. For example, we may search for all the values of x that solve an equation such that $0 \leqslant x \leqslant 2\pi$. Although it is certainly possible to write a general expression using a parameter (i.e. the general solution) that specifies the infinite values that solve a trigonometric equation, it is not required for this course. A solution interval will always be given, as in the example below.

Example 15

Find the exact solution(s) to the equation $\sin x = \frac{1}{2}$ for $0 \leqslant x \leqslant 2\pi$.

Solution

Recalling the definition of the sine function, this equation can be interpreted as asking for the length, x, of arcs along the unit circle that have a terminal point with a y-coordinate equal to $\frac{1}{2}$. We know, from Section 6.2, that arc lengths of $\frac{\pi}{6}$ and $\frac{5\pi}{6}$ have terminal points with y-coordinates of $\frac{1}{2}$. There are clearly an infinite number of arcs – both positive and negative – that will terminate at the same points which can be written as $x = \frac{\pi}{6} + k \cdot 2\pi$ and $x = \frac{5\pi}{6} + k \cdot 2\pi$, $k \in \mathbb{Z}$. However, we are only asked for the solutions in the interval $0 \leqslant x \leqslant 2\pi$. Therefore, $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$.



Another way to see that the equation $\sin x = \frac{1}{2}$ has infinitely many solutions is to graph the equations $y = \sin x$ and $y = \frac{1}{2}$ (Figure 6.28) and search for intersection points, i.e. where the two equations are equal.

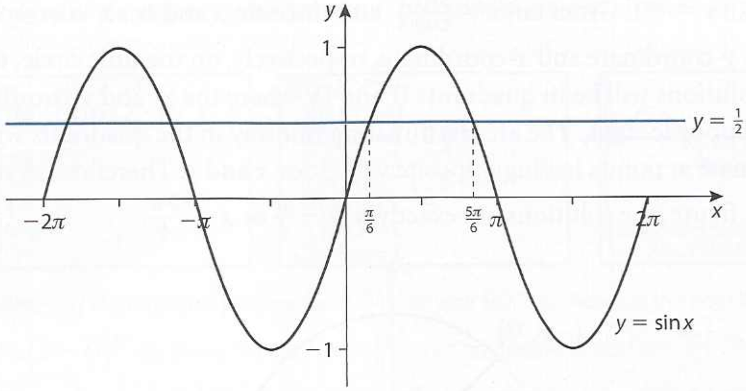
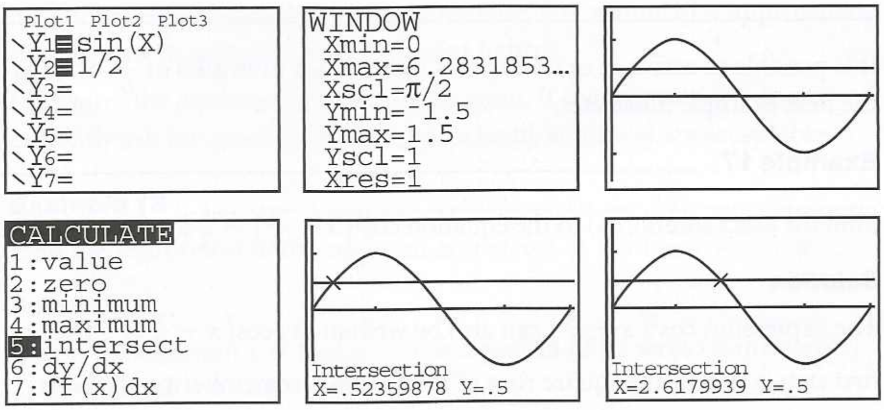


Figure 6.28

The graphs of the two equations will intersect repeatedly as they extend indefinitely in both directions.

Your GDC can be a very effective tool for searching for solutions graphically. However, it can be limited when exact solutions are requested. The sequence of GDC images below show a graphical solution for the equation in Example 15.



The GDC gives two solutions in the interval $0 \leqslant x \leqslant 2\pi$ as $x = 0.523\,598\,78$ and $x = 2.617\,9939$. These values are approximations (to 8 significant figures) of two irrational numbers: $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$. Therefore, if you wish, or need, to find exact solutions, you will need to remember the trigonometric function values for the multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ (see Figures 6.18 and 6.19 in Section 6.2).

Example 16

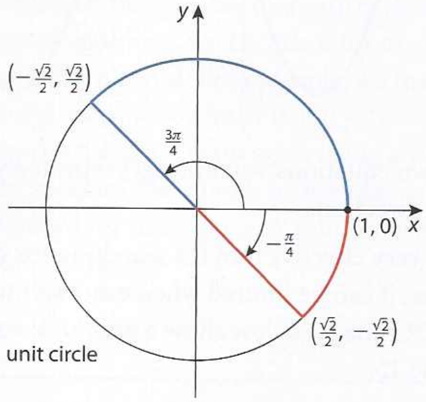
Find the exact solution(s) to the equation $\tan(x) + 1 = 0$ for $-\pi \leqslant x < \pi$.

Solution

It's important to note that the solution interval is different than for Example 15. The possible values of x include negative values (from 0 to $-\pi$) and positive values (from 0 to π). With respect to the unit circle, the solutions will correspond to points in any of the quadrants (as for Example 15) but points in quadrants III and IV will correspond to arcs rotating clockwise (negative direction). Solutions to this equation are values of x such

• **Hint:** The expression $\tan x + 1$ is not equivalent to $\tan(x + 1)$. In the first expression, x alone is the argument of the function, and in the second expression, $x + 1$ is the argument of the function. It is a good habit to use brackets to make it absolutely clear what is, or is not, the argument of a function. For example, there is no ambiguity if $\tan x + 1$ is written as $\tan(x) + 1$, or as $1 + \tan x$.

that $\tan x = -1$. Given $\tan x = \frac{\sin x}{\cos x}$ and since $\sin x$ and $\cos x$ correspond to the y-coordinate and x-coordinate, respectively, on the unit circle, then any solutions will be in quadrants II and IV where the x- and y-coordinates have opposite signs. The arcs terminating midway in the quadrants will terminate at points having opposite values for x and y. Therefore, as shown in the figure, the solutions are exactly $x = -\frac{\pi}{4}$ or $x = \frac{3\pi}{4}$.



It is possible to arrive at exact answers that are not multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$, as the next example illustrates.

Example 17

Find the exact solution(s) to the equation $\cos^2\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$ for $0 \leqslant x \leqslant 2\pi$.

Solution

The expression $\cos^2\left(x - \frac{\pi}{3}\right)$ can also be written as $\left[\cos\left(x - \frac{\pi}{3}\right)\right]^2$. The first step is to take the square root of both sides – remembering that every positive number has two square roots – which gives $\cos\left(x - \frac{\pi}{3}\right) = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$. All of the odd integer multiples of $\frac{\pi}{4}\left(\ldots -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}, \ldots\right)$ have a cosine equal to either $\frac{\sqrt{2}}{2}$ or $-\frac{\sqrt{2}}{2}$.

That is, $x - \frac{\pi}{3} = \frac{\pi}{4} + k \cdot \frac{\pi}{2}$. Now, solve for x.

$x = \frac{\pi}{4} + \frac{\pi}{3} + k \cdot \frac{\pi}{2} = \frac{7\pi}{12} + k \cdot \frac{6\pi}{12}$. The last step is to substitute in different integer values for k to generate all the possible values for x so that $0 \leqslant x \leqslant 2\pi$.

When $k = 0$: $x = \frac{7\pi}{12}$; when $k = 1$: $x = \frac{7\pi}{12} + \frac{6\pi}{12} = \frac{13\pi}{12}$;

when $k = 2$: $x = \frac{7\pi}{12} + \frac{12\pi}{12} = \frac{19\pi}{12}$;

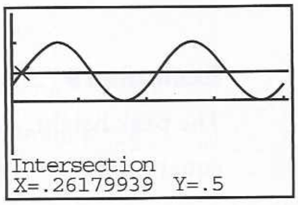
when $k = 3$: $x = \frac{7\pi}{12} + \frac{18\pi}{12} = \frac{25\pi}{12}$; … however, $\frac{25\pi}{12} > 2\pi$…but

when $k = -1$: $x = \frac{7\pi}{12} - \frac{6\pi}{12} = \frac{\pi}{12}$. There are four exact solutions in the interval $0 \leqslant x \leqslant 2\pi$ and they are: $x = \frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}$ or $\frac{19\pi}{12}$.

• **Hint:** Check the solutions to trigonometric equations with your GDC. The sequence of GDC images here verifies that $x = \frac{\pi}{12}$ is the first solution to the equation in Example 17.

```
Plot1 Plot2 Plot3
\Y1■(cos(X-π/3))
2
\Y2■1/2
\Y3=
\Y4=
\Y5=
\Y6=
```

```
WINDOW
 Xmin=0
 Xmax=6.283
 Xscl=1.570
 Ymin=-1.5
 Ymax=1.5
 Yscl=1
 Xres=1
```




```
π/12
              .2617993878
```

When entering the equation $y = \cos^2\left(x - \frac{\pi}{3}\right)$ into your GDC (as shown in the first GDC image), you will have to enter it in the form $y = \left[\cos\left(x - \frac{\pi}{3}\right)\right]^2$. Be aware that $\cos^2\left(x - \frac{\pi}{3}\right)$ is not equivalent to $\cos\left(x - \frac{\pi}{3}\right)^2$. The expression $\cos\left(x - \frac{\pi}{3}\right)^2$ indicates that the quantity $x - \frac{\pi}{3}$ is squared first and then the cosine of the resulting value is found. However, the expression $y = \cos^2\left(x - \frac{\pi}{3}\right)$ indicates that the cosine of $x - \frac{\pi}{3}$ is found first and then that value is squared.

Graphical solutions to trigonometric equations

If exact solutions are not required then a graphical solution using your GDC is a very effective way to find approximate solutions to trigonometric equations. Unless instructed to do otherwise, you should give approximate solutions to an accuracy of 3 significant figures.

Let's solve the equation in Example 16 again. If the instructions do not explicitly ask for exact solutions, approximate solutions are acceptable.

Example 18

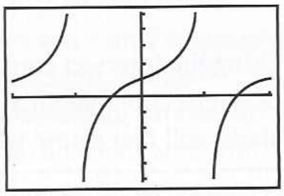
Find the solution(s) to the equation $\tan(x) + 1 = 0$ for $-\pi \leqslant x < \pi$.

Solution

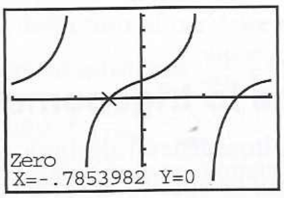
Graph the equation $y = \tan(x) + 1$ and find all of its zeros (x-intercepts) in the interval $-\pi \leqslant x < \pi$.

```
Plot1 Plot2 Plot3
\Y1■tan(X)+1
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

```
WINDOW
 Xmin=-3.141592…
 Xmax=3.1415926…
 Xscl=1.5707963…
 Ymin=-5
 Ymax=5
 Yscl=1
 Xres=1
```



```
CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
```




```
X
            -.7853981634
Ans+π
              2.35619449
```

This sequence of GDC images indicates an approximate solution $x \approx -0.785$ between 0 and $-\pi$. Since we know that the period of $y = \tan x + 1$ is π (same as for $y = \tan x$), we can simply add π to this first solution to find the one between 0 and π, as shown in the final GDC image. Therefore, two solutions for x in the interval $-\pi \leqslant x < \pi$ are $x \approx -0.785$ and $x \approx 2.36$ (accuracy to 3 significant figures).

A graphical approach is effective and appropriate when it is not possible, or very difficult, to find exact solutions.

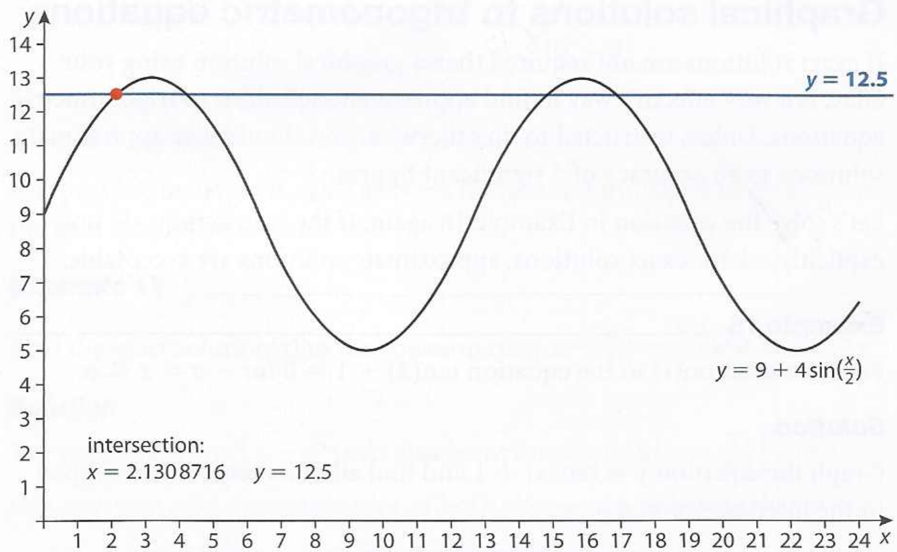
Example 19

The peak height, h metres, of ocean waves during a storm is given by the equation $h = 9 + 4\sin\left(\frac{t}{2}\right)$, where t is the number of hours after midnight.

A tsunami alarm is triggered when the peak height goes above 12.5 metres. Find the value of t when the alarm first sounds.

Solution

Graph the equations $y = 9 + 4\sin\left(\frac{x}{2}\right)$ and $y = 12.5$ and find the first point of intersection for $x > 0$.



Using the Intersect command on the GDC indicates that the first point of intersection has an x-coordinate of approximately 2.13. Therefore, the alarm will first sound when $t \approx 2.13$ hours.

Analytic solutions to trigonometric equations

In this section, we will see how general algebraic techniques and trigonometric identities can be applied to solve trigonometric equations. An analytical approach requires you to devise a solution strategy utilizing algebraic methods that you have applied to other types of equations – such as quadratic equations. Often, trigonometric equations that demand an analytic approach will result in exact solutions, but not always. Although our approach for equations in this section focuses on algebraic techniques, it is important to use graphical methods to support or confirm our analytical solutions.

Example 20

Solve $2\sin^2 x - \sin x = 0$ for $-\pi \leqslant x \leqslant \pi$.

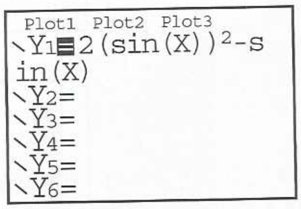
Solution

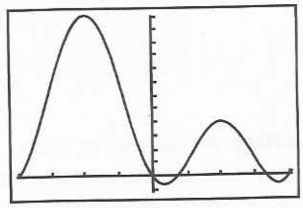
We can factorize and apply the rule that if $a \cdot b = 0$ then either $a = 0$ or $b = 0$.

$2\sin^2 x - \sin x = 0 \Rightarrow \sin x(2\sin x - 1) = 0 \Rightarrow \sin x = 0$ or $\sin x = \frac{1}{2}$

For $\sin x = 0$: $x = -\pi, 0, \pi$; for $\sin x = \frac{1}{2}$: $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

Therefore, $x = -\pi, 0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi$.





• **Hint:** Although exact answers were not demanded in Example 20, given our knowledge of the unit circle and familiarity with the sine of common values (i.e. multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$), we are able to give exact answers without any difficulty. It would have been acceptable to give approximate solutions, but it is worth recognizing that this would have required considerable more effort than providing exact solutions. Entering and graphing the equation $y = 2\sin^2 x - \sin x$ on your GDC (see GDC images) would not be the most efficient or appropriate solution method, but, if sufficient time is available, it is an effective way to confirm your exact solutions.

The next example illustrates how the application of a trigonometric identity can be helpful to rewrite an equation in a way that allows us to solve it algebraically.

Example 21

Solve $3\sin x + \tan x = 0$ for $0 \leqslant x \leqslant 2\pi$.

Solution

Since the structure of this equation is such that an expression is set equal to zero, it would be nice to be able to use the same algebraic technique as the previous example – that is, factorize and solve for when each factor is zero. However, it is not possible to factorize the expression $3\sin x + \tan x$, and rewriting the equation as $3\sin x = -\tan x$ does not help. Are there any expressions in the equation for which we can substitute an equivalent expression that will make the equation accessible to an algebraic solution? We do not have any equivalent expressions for $\sin x$, but we do have an identity for $\tan x$. From the definition of $\tan x$, we know that $\tan x = \frac{\sin x}{\cos x}$.

Let's see what happens when we substitute $\frac{\sin x}{\cos x}$ for $\tan x$.

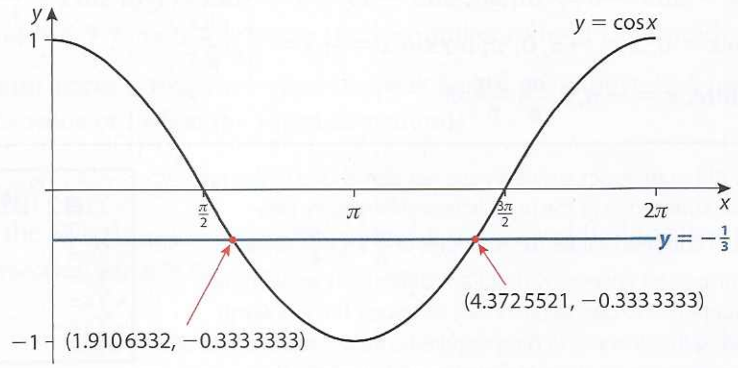
$$3\sin x + \tan x = 0 \Rightarrow 3\sin x + \frac{\sin x}{\cos x} = 0$$

Now, multiply both sides by $\cos x$ while recognising that $\cos x \neq 0$ $(x \neq \frac{\pi}{2} + k \cdot \pi, k \in \mathbb{Z})$.

$$3\sin x + \frac{\sin x}{\cos x} = 0 \Rightarrow 3\sin x\cos x + \sin x = 0 \Rightarrow \sin x(3\cos x + 1) = 0 \Rightarrow$$
$$\sin x = 0 \text{ or } \cos x = -\frac{1}{3}$$

For $\sin x = 0$: $x = 0, \pi, 2\pi$.

We know that $(1, 0)$ and $(-1, 0)$ are the points on the unit circle that correspond to $\sin x = 0$ giving the three exact solutions above. Although we know that the points on the unit circle that correspond to $\cos x = -\frac{1}{3}$ will be in the second and third quadrants, we do not know their exact coordinates. So, we will need to use our GDC to find approximate solutions to $\cos x = -\frac{1}{3}$ for $0 \leqslant x \leqslant 2\pi$.



Thus, for $\cos x = -\frac{1}{3}$: $x \approx 1.91$ or $x \approx 4.37$ (three significant figures). Therefore, the full solution set for the equation is $x = 0, \pi, 2\pi$; $x \approx 1.91, 4.37$.

• **Hint:** A strategy that often proves fruitful is to try and rewrite a trigonometric equation in terms of just one trigonometric function. If that is not possible, try and rewrite it in terms of only the sine and cosine functions. This strategy was used in Example 21.

Trigonometric identities

As Example 21 illustrated, sometimes an analytical method for solving a trigonometric equation relies on a trigonometric identity providing a useful substitution. There are a few trigonometric identities, other than $\tan x = \frac{\sin x}{\cos x}$, required for this course which can be used to help simplify trigonometric expressions and solve equations.

At the start of this section, it was stated that the equation $\sin^2 x + \cos^2 x = 1$ is an identity; that is, it's true for all possible values of x. Let's prove that this is the case.

Recall from Section 6.1 that the equation for the unit circle is $x^2 + y^2 = 1$. That is, the coordinates (x, y) of any point on the circle will satisfy the equation $x^2 + y^2 = 1$. Also, in Section 6.2, we learned that the sine and cosine functions are defined in terms of the coordinates of the terminal point of an arc on the unit circle starting at (1, 0), as shown in Figure 6.29. If t is any real number that is the length of an arc on the unit circle that terminates at (x, y), then $x = \cos t$ and $y = \sin t$. Substituting directly into the equation for the circle gives $\sin^2 t + \cos^2 t = 1$. As mentioned in Section 6.3, the convention is to use x to denote the domain variable rather than t. Therefore, the equation $\sin^2 x + \cos^2 x = 1$ is true for any real number x.

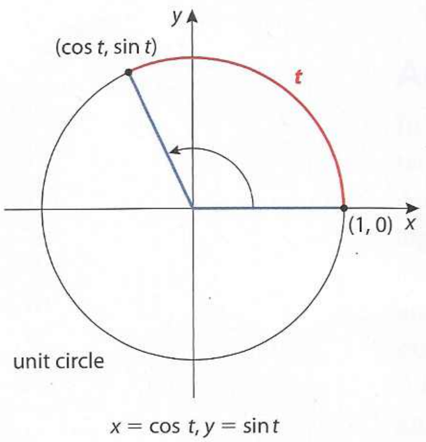


Figure 6.29

The Pythagorean identities for sine and cosine

The following equations are true for all real numbers x:

$$\sin^2 x + \cos^2 x = 1 \qquad \sin^2 x = 1 - \cos^2 x \qquad \cos^2 x = 1 - \sin^2 x$$

Another useful set of trigonometric identities are referred to as the **double angle identities** because they are equations involving $\sin 2x$ and $\cos 2x$. As discussed back in Section 6.1, the argument of a trigonometric function (x in $\sin x$, θ in $\cos\theta$) can be interpreted as an angle (in degrees or radians), or as just a real number. Even though these identities are called double *angle* identities they apply for either interpretation.

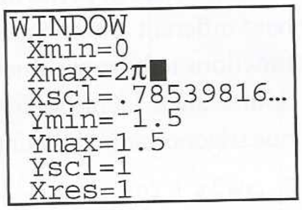
Double angle identities for sine and cosine

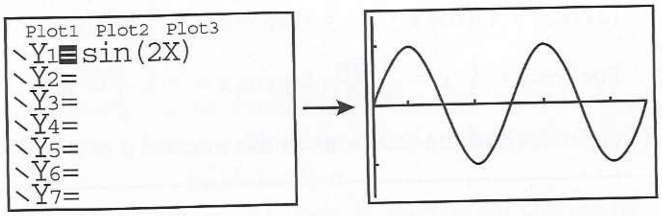
The following equations are true for all real numbers x:

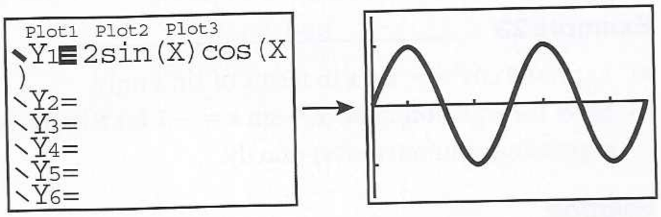
$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \begin{cases} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases}$$

It is quite easy to verify the double angle identities by means of graphical analysis on your GDC.







The GDC screen images shown here illustrate that $\sin 2x$ is equivalent to $2\sin x\cos x$. Use your GDC to verify that $\cos 2x$ is equivalent to $\cos^2 x - \sin^2 x$. Once the identity $\cos 2x = \cos^2 x - \sin^2 x$ is established we can use one of the Pythagorean identities to rewrite it in terms of only sine or cosine; thus, establishing the other two double angle identities for cosine.

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = \cos^2 x - (1 - \cos^2 x) \qquad \text{substitute } 1 - \cos^2 x \text{ for } \sin^2 x$$

$$\cos 2x = 2\cos^2 x - 1$$

Similar steps can be performed to show that $\cos 2x = 1 - 2\sin^2 x$.

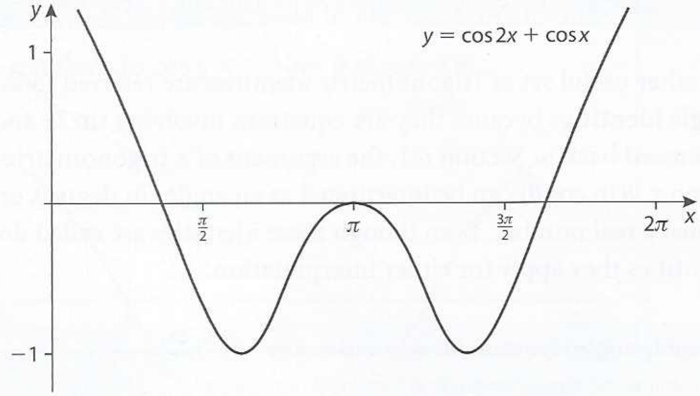
Now let's see how these identities can help us with algebraic solutions of trigonometric equations.

The identity $\sin^2 x + \cos^2 x = 1$ is often referred to as a Pythagorean identity because, as we will see in the other chapter on trigonometry, $\sin x$ and $\cos x$ can represent the legs of a right-angled triangle with a hypotenuse equal to one. Substituting into the Pythagorean theorem gives $\sin^2 x + \cos^2 x = 1$.

Example 22

Solve the equation $\cos 2x + \cos x = 0$ for $0 \leqslant x \leqslant 2\pi$.

Solution



Taking an initial look at the graph of $y = \cos 2x + \cos x$ suggests that there are possibly three solutions in the interval $x \in [0, 2\pi]$. Although the expression $\cos 2x + \cos x$ contains terms with only the cosine function, it is not possible to perform any algebraic operations on them because they have different arguments. In order to solve algebraically, we need both cosine functions to have arguments of x (rather than $2x$). There are three different double angle identities for $\cos 2x$. It is best to have the equation in terms of one trigonometric function, so we choose to substitute $2\cos^2 x - 1$ for $\cos 2x$.

$$\cos 2x + \cos x = 0 \Rightarrow 2\cos^2 x - 1 + \cos x = 0 \Rightarrow 2\cos^2 x + \cos x - 1 = 0$$

$$(2\cos x - 1)(\cos x + 1) = 0 \Rightarrow \cos x = \frac{1}{2} \text{ or } \cos x = -1$$

For $\cos x = \frac{1}{2}$: $x = \frac{\pi}{3}, \frac{5\pi}{3}$; for $\cos x = -1$: $x = \pi$.

Therefore, all of the solutions in the interval $0 \leqslant x \leqslant 2\pi$ are: $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

Example 23

a) Express $2\cos^2 x + \sin x$ in terms of $\sin x$ only.
b) Solve the equation $2\cos^2 x + \sin x = -1$ for x in the interval $0 \leqslant x \leqslant 2\pi$, expressing your answer(s) exactly.

Solution

a) $2\cos^2 x + \sin x = 2(1 - \sin^2 x) + \sin x$ using Pythagorean identity $\cos^2 x = 1 - \sin^2 x$

$= 2 - 2\sin^2 x + \sin x$

b) $2\cos^2 x + \sin x = -1$

$2 - 2\sin^2 x + \sin x = -1$ substitute result from a)

$2\sin^2 x - \sin x - 3 = 0$ [alternatively: let $\sin x = y$, then $2y^2 - y - 3 = 0$]

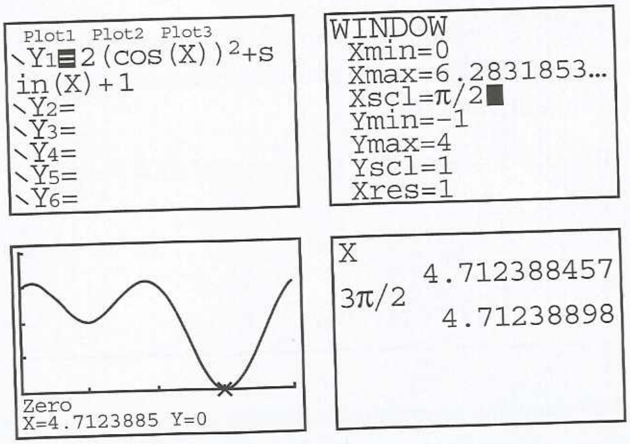
$(2\sin x - 3)(\sin x + 1) = 0$ factorize [alt: $(2y - 3)(y + 1) = 0$]

$\sin x = \frac{3}{2}$ or $\sin x = -1$ [alt: $y = \frac{3}{2}$ or $y = -1 \Rightarrow \sin x = \frac{3}{2}$ or $\sin x = -1$]

For $\sin x = \frac{3}{2}$: no solution because $\frac{3}{2}$ is not in the range of the sine function.

For $\sin x = -1$: $x = \frac{3\pi}{2}$. Therefore, only one solution in $0 \leqslant x \leqslant 2\pi$: $x = \frac{3\pi}{2}$.

Use your GDC to check this result by rewriting $2\cos^2 x + \sin x = -1$ as $2\cos^2 x + \sin x + 1 = 0$ and then graph $y = 2\cos^2 x + \sin x + 1$; confirming a single zero at $x = \frac{3\pi}{2}$ in the interval $x \in [0, 2\pi]$.



Example 24

Solve the equation $2\sin 2x = 3\cos x$ for $0 \leqslant x \leqslant \pi$.

Solution

$2\sin 2x = 3\cos x$

$2(2\sin x\cos x) = 3\cos x$ using double angle identity for sine

$4\sin x\cos x = 3\cos x$ do *not* divide by $\cos x$ as solution(s) may be eliminated

$4\sin x\cos x - 3\cos x = 0$ set equal to zero to prepare for solving by factorization

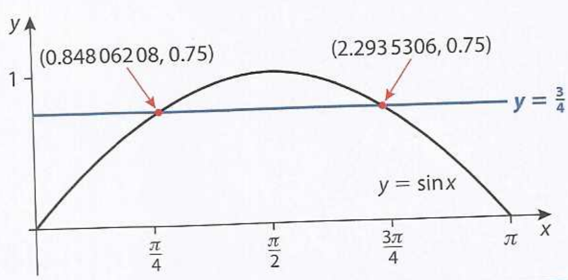
$\cos x(4\sin x - 3) = 0$ factorize

$\cos x = 0$ or $\sin x = \frac{3}{4}$

For $\cos x = 0$: $x = \frac{\pi}{2}$.

For $\sin x = \frac{3}{4}$: $x \approx 0.848$ or 2.29.

Approximate solutions found using Intersect command on GDC. All solutions in the interval $0 \leqslant x \leqslant \pi$ are: $x = \frac{\pi}{2}$ and $x \approx 0.848, 2.29$.



The final example illustrates how trigonometric identities can be applied to find exact values for trigonometric expressions.

Example 25

Given that $\cos x = \frac{1}{4}$ and $0 < x < \frac{\pi}{2}$, find the *exact* values of

a) $\sin x$ b) $\sin 2x$.

Solution

a) Given $0 < x < \frac{\pi}{2}$ it follows that $\sin x > 0$, because the arc with length x will terminate in the first quadrant. The Pythagorean identity is useful when relating $\sin x$ and $\cos x$.

$$\sin^2 x = 1 - \cos^2 x \Rightarrow \sin x = \sqrt{1 - \cos^2 x} \Rightarrow \sin x = \sqrt{1 - \left(\frac{1}{4}\right)^2}$$
$$= \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$$

b) $\sin 2x = 2 \sin x \cos x = 2\left(\frac{\sqrt{15}}{4}\right)\left(\frac{1}{4}\right) = \frac{\sqrt{15}}{8}$

Summary of fundamental trigonometric identities

Definition of tangent function:	$\tan x = \frac{\sin x}{\cos x}$	
Odd/even function identities:	$\sin(-x) = -\sin x$	$\cos(-x) = \cos x$
Co-function identities:	$\sin\left(\frac{\pi}{2} - x\right) = \cos x$	$\cos\left(\frac{\pi}{2} - x\right) = \sin x$
Pythagorean identities:	$\sin^2 x + \cos^2 x = 1$	
	$\sin^2 x = 1 - \cos^2 x$	$\cos^2 x = 1 - \sin^2 x$
Double angle identities:	$\sin 2x = 2 \sin x \cos x$	
	$\cos 2x = \cos^2 x - \sin^2 x$	
	$\cos 2x = 2\cos^2 x - 1$	$\cos 2x = 1 - 2\sin^2 x$

Exercise 6.4

In questions 1–10, find the exact solution(s) for $0 \leq x \leq 2\pi$. Verify your solution(s) with your GDC.

1 $\cos x = \frac{1}{2}$

2 $2 \sin x + 1 = 0$

3 $1 - \tan x = 0$

4 $\sqrt{3} = 2 \sin x$

5 $2 \sin^2 x = 1$

6 $4 \cos^2 x = 3$

7 $\tan^2 x - 1 = 0$

8 $4 \cos^2 x = 1$

9 $\tan x(\tan x + 1) = 0$

10 $\sin x \cos x = 0$

In questions 11–16, use your GDC to find approximate solution(s) for $0 \leq x \leq 2\pi$. Express solutions accurate to 3 significant figures.

11 $\sin x = 0.4$

12 $3 \cos x + 1 = 0$

13 $\tan x = 2$

14 $\sin 2x = 0.85$

15 $\cos(x - 1) = -0.38$

16 $3 \tan x = 10$

In questions 17–20, given that k is any integer, list all of the possible values for x that are in the specified interval.

17 $x = \frac{\pi}{2} + k \cdot \pi, -3\pi \leqslant x \leqslant 3\pi$

18 $x = \frac{\pi}{6} + k \cdot 2\pi, -2\pi \leqslant x \leqslant 2\pi$

19 $x = \frac{7\pi}{12} + k \cdot \pi, 0 \leqslant x \leqslant 2\pi$

20 $x = \frac{\pi}{4} + k \cdot \frac{\pi}{4}, 0 \leqslant x \leqslant 2\pi$

In questions 21–24, find the exact solutions for $0 \leqslant x \leqslant 2\pi$.

21 $\cos\left(x - \frac{\pi}{6}\right) = -\frac{1}{2}$

22 $\tan(x + \pi) = 1$

23 $\sin 2x = \frac{\sqrt{3}}{2}$

24 $\sin^2\left(x + \frac{\pi}{2}\right) = \frac{3}{4}$

25 The number, N, of empty birds' nests in a park is approximated by the function $N = 74 + 42\sin\left(\frac{\pi}{12}t\right)$, where t is the number of hours after midnight. Find the value of t when the number of empty nests first equals 90. Approximate the answer to 1 decimal place.

26 In Edinburgh, the number of hours of daylight on day D is modelled by the function $H = 12 + 7.26\sin\left[\frac{2\pi}{365}(D - 80)\right]$, where D is the number of days after December 31 (e.g. January 1 is $D = 1$, January 2 is $D = 2$, and so on). Do not use your GDC on part a).

a) Which days of the year have 12 hours of daylight?

b) Which days of the year have about 15 hours of daylight?

c) How many days of the year have more than 17 hours of daylight?

In questions 27–34, solve the equation for the stated solution interval. Find exact solutions, if possible. Otherwise, give solutions to 3 significant figures. Verify solutions with your GDC.

27 $2\cos^2 x + \cos x = 0; 0 \leqslant x \leqslant 2\pi$

28 $2\sin^2 x - \sin x - 1 = 0; 0 \leqslant x \leqslant 2\pi$

29 $2\cos x + \sin 2x = 0; -\pi \leqslant x \leqslant \pi$

30 $2\sin x = \cos 2x; -\pi \leqslant x \leqslant \pi$

31 $\tan^2 x - \tan x = 2; -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

32 $\sin^2 x = \cos^2 x; 0 \leqslant x \leqslant \pi$

33 $2\sin^2 x + 3\cos x - 3 = 0; 0 \leqslant x \leqslant 2\pi$

34 $2\sin x = 3\cos x; 0 \leqslant x \leqslant 2\pi$

35 Given that $\sin x = \frac{3}{5}$ and $0 < x < \frac{\pi}{2}$, find the *exact* values of

a) $\cos x$ b) $\cos 2x$ c) $\sin 2x$.

36 Given that $\cos x = -\frac{2}{3}$ and $\frac{\pi}{2} < x < \pi$, find the *exact* values of

a) $\sin x$ b) $\sin 2x$ c) $\cos 2x$.

Practice questions

1 A toy on an elastic string is attached to the top of a doorway. It is pulled down and released, allowing it to bounce up and down. The length of the elastic string, L centimetres, is modelled by the function $L = 110 + 25\cos(2\pi t)$, where t is time in seconds after release.

a) Find the length of the elastic string after 2 seconds.

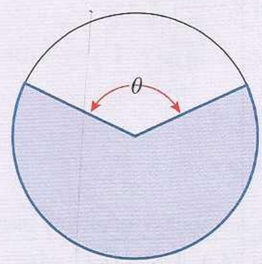
b) Find the minimum length of the string.

c) Find the first time after release that the string is 85 cm.

d) What is the period of the motion?

2 Find the exact solution(s) to the equation $2\sin^2 x - \cos x + 1 = 0$ for $0 \leqslant x \leqslant 2\pi$.

3 The diagram shows a circle of radius 6 cm. The perimeter of the shaded sector is 25 cm. Find the radian measure of the angle θ.



4 Consider the two functions $f(x) = \cos 4x$ and $g(x) = \cos\left(\frac{x}{2}\right)$.

a) Write down: **(i)** the minimum value of the function f

(ii) the period of g.

b) For the equation $f(x) = g(x)$, find the number of solutions in the interval $0 \leqslant x \leqslant \pi$.

5 A reflector is attached to the spoke of a bicycle wheel. As the wheel rolls along the ground, the distance, d centimetres, that the reflector is above the ground after t seconds is modelled by the function

$$d = p + q\cos\left(\frac{2\pi}{m}t\right), \text{ where } p, q \text{ and } m \text{ are constants.}$$

The distance d is at a maximum of 64 cm at $t = 0$ seconds and at $t = 0.5$ seconds, and is at a minimum of 6 cm at $t = 0.25$ seconds and at $t = 0.75$ seconds. Write down the value of:

a) p **b)** q **c)** m.

6 Find all solutions to $1 + \sin 3x = \cos(0.25x)$ such that $x \in [0, \pi]$.

7 Find all solutions to both trigonometric equations in the interval $x \in [0, 2\pi]$. Express the solutions exactly.

a) $2\cos^2 x + 5\cos x + 2 = 0$ **b)** $\sin 2x - \cos x = 0$

8 The value of x is in the interval $\frac{\pi}{2} < x < \pi$ and $\cos^2 x = \frac{8}{9}$. Without using your GDC, find the exact values for the following:

a) $\sin x$ **b)** $\cos 2x$ **c)** $\sin 2x$

9 The depth, d metres, of water in a harbour varies with the tides during each day. The first high (maximum) tide after midnight occurs at 5:00 a.m. with a depth of 5.8 m. The first low (minimum) tide occurs at 10:30 a.m. with a depth of 2.6 m.

a) Find a trigonometric function that models the depth, d, of the water t hours after midnight.

b) Find the depth of the water at 12 noon.

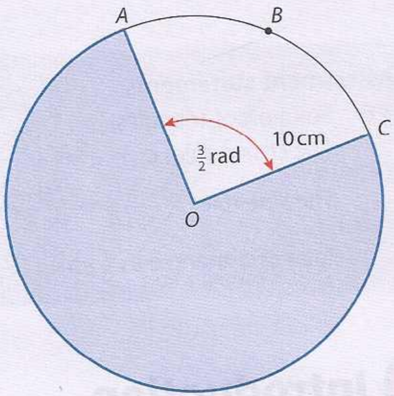
c) A large boat needs at least 3.5 m of water to dock in the harbour. During what time interval after 12 noon can the boat dock safely?

10 Solve the equation $\tan^2 x + 2\tan x - 3 = 0$ for $0 \leqslant x \leqslant \pi$. Give solutions exactly, if possible. Otherwise, give solutions to 3 significant figures.

11 The following diagram shows a circle of centre O and radius 10 cm. The arc ABC subtends an angle of $\frac{3}{2}$ radians at the centre O.

a) Find the length of the arc ACB.

b) Find the area of the shaded region.



12 Consider the function $f(x) = \frac{5}{2}\cos\left(2x - \frac{\pi}{2}\right)$. For what values of k will the equation $f(x) = k$ have no solutions?

13 A portion of the graph of $y = k + a\sin x$ is shown below. The graph passes through the points (0, 1) and $\left(\frac{3\pi}{2}, 3\right)$. Find the value of k and a.