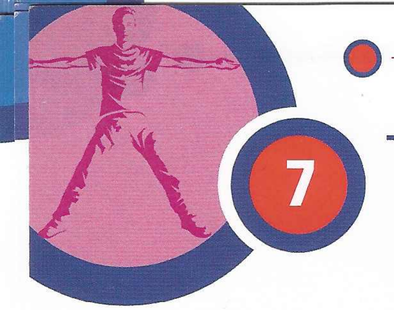


7 Triangle Trigonometry

Assessment statements

3.6 Solution of triangles.
The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$.
The sine rule: $= \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.
Area of a triangle as $\frac{1}{2}ab\sin C$.

Introduction

In this chapter, we approach trigonometry from a **right triangle** perspective where trigonometric functions will be defined in terms of the **ratios of sides of a right triangle**. Over two thousand years ago, the Greeks developed trigonometry to make helpful calculations for surveying, navigating, building and other practical pursuits. Their calculations were based on the angles and lengths of sides of a right triangle. The modern development of trigonometry, based on the length of an arc on the unit circle, was covered in the previous chapter. We begin a more classical approach by introducing some terminology regarding right triangles.

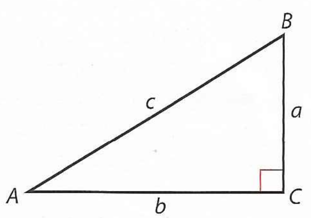


Figure 7.1

● **Hint:** In IB notation, [AC] denotes the line segment connecting points A and C. The notation AC represents the *length* of this line segment. Also, the notation $A\hat{B}C$ denotes the angle with its vertex at point B, with one side of the angle containing the point A and the other side containing point C.

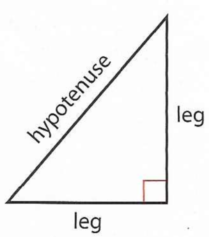


Figure 7.2

7.1 Right triangles and trigonometric functions

Right triangles

The conventional notation for triangles is to label the three vertices with capital letters, for example A, B and C. The same capital letters can be used to represent the measure of the angles at these vertices. However, we will often use a Greek letter, such as α (alpha), β (beta) or θ (theta) to do so. The corresponding lower-case letters, a, b and c, represent the lengths of the sides opposite the vertices. For example, b represents the length of the side opposite angle B, that is, the line segment AC, or $[AC]$ (Figure 7.1).

In a right triangle, the longest side is opposite the right angle (i.e. measure of 90°) and is called the **hypotenuse**, and the two shorter sides adjacent to the right angle are often called the **legs** (Figure 7.2). Because the sum of the three angles in any triangle in plane geometry is 180°, then the two non-right angles are both **acute angles** (i.e. measure between 0 and 90 degrees). It also follows that the two acute angles in a right triangle are a pair of **complementary angles** (i.e. have a sum of 90°).

Trigonometric functions of an acute angle

We can use properties of similar triangles and the definitions of the sine, cosine and tangent functions from Chapter 6 to define these functions in terms of the sides of a right triangle.

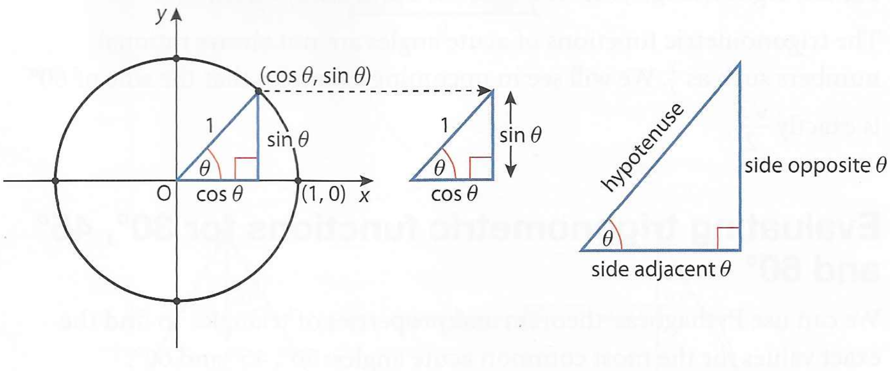


Figure 7.3

The right triangles shown in Figure 7.3 are **similar triangles** because corresponding angles have equal measure – each has a right angle and an acute angle of measure θ. It follows that the ratios of corresponding sides are equal, allowing us to write the following three proportions involving the sine, cosine and tangent of the acute angle θ.

$$\frac{\sin\theta}{1} = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \frac{\cos\theta}{1} = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta} = \frac{\text{opposite}}{\text{adjacent}}$$

The definitions of the trigonometric functions in terms of the sides of a right triangle follow directly from these three equations.

Right triangle definition of the trigonometric functions

Let θ be an **acute angle** of a right triangle, then the sine, cosine and tangent functions of the angle θ are defined as the following ratios in the right triangle:

$$\sin\theta = \frac{\text{side opposite angle }\theta}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{side adjacent angle }\theta}{\text{hypotenuse}}$$

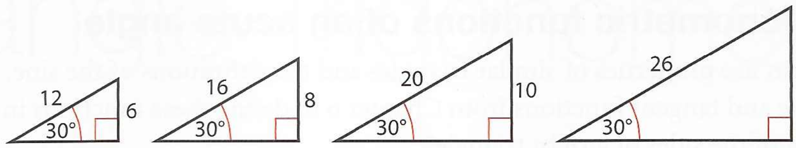
$$\tan\theta = \frac{\text{side opposite angle }\theta}{\text{side adjacent angle }\theta}$$

It follows that the sine, cosine and tangent of an acute angle are positive.

It is important to understand that properties of similar triangles are the foundation of right triangle trigonometry. Regardless of the size (i.e. lengths of sides) of a right triangle, so long as the angles do not change, the ratio of any two sides in the right triangle will remain *constant*. All the right triangles in Figure 7.4 have an acute angle with a measure of 30° (thus, the other acute angle is 60°). For each triangle, the ratio of the side opposite the 30° angle to the hypotenuse is exactly $\frac{1}{2}$. In other words, the sine of 30° is always $\frac{1}{2}$. This agrees with results from the previous chapter knowing that an angle of 30° is equivalent to $\frac{\pi}{6}$ in radian measure.

Thales of Miletus (circa 624–547) was the first of the Seven Sages, or wise men of ancient Greece, and is considered by many to be the first Greek scientist, mathematician and philosopher. Thales visited Egypt and brought back knowledge of astronomy and geometry. According to several accounts, Thales, with no special instruments, determined the height of Egyptian pyramids. He applied formal geometric reasoning. Diogenes Laertius, a 3rd-century biographer of ancient Greek philosophers, wrote: 'Hieronymus says that [Thales] even succeeded in measuring the pyramids by observation of the length of their shadow at the moment when our shadows are equal to our own height.' Thales used the geometric principle that the ratios of corresponding sides of similar triangles are equal.

Figure 7.4



For any right triangle, the sine ratio for 30° is always $\frac{1}{2}$: $\sin 30° = \frac{1}{2}$.

The trigonometric functions of acute angles are not always rational numbers such as $\frac{1}{2}$. We will see in upcoming examples that the sine of 60° is exactly $\frac{\sqrt{3}}{2}$.

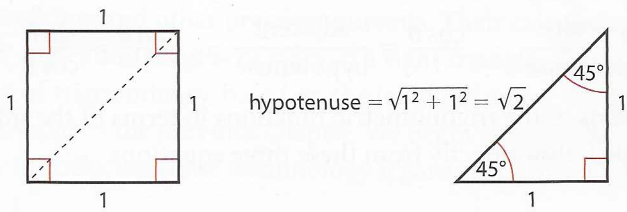
Evaluating trigonometric functions for 30°, 45° and 60°

We can use Pythagoras' theorem and properties of triangles to find the exact values for the most common acute angles: 30°, 45° and 60°.

Example 1

Find the values of sin 45°, cos 45° and tan 45°.

Solution



Consider a square with each side equal to one unit. Draw a diagonal of the square, forming two isosceles right triangles. From geometry, we know that the diagonal will bisect each of the two right angles forming two isosceles right triangles, each with two acute angles of 45°. The isosceles right triangles have legs of length one unit and, from Pythagoras' theorem, a hypotenuse of exactly $\sqrt{2}$ units. The trigonometric functions are then calculated as follows:

$$\sin 45° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$ [multiplying by $\frac{\sqrt{2}}{\sqrt{2}}$ to rationalize the denominator]

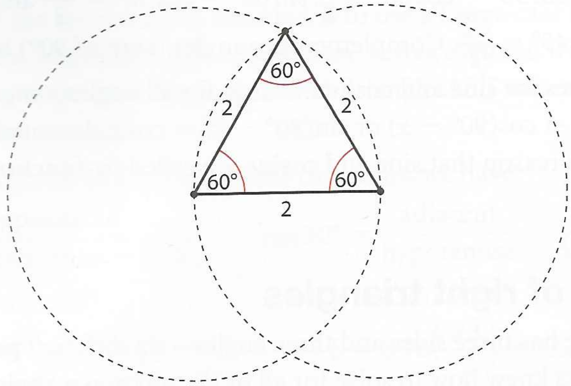
$$\cos 45° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

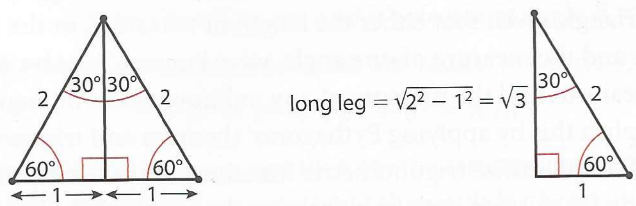
$$\tan 45° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1$$

Example 2

Find the values of the sine, cosine and tangent functions for 30° and 60°.

Solution





Start with a line segment of length two units. Using each endpoint as a centre and the segment as a radius, construct two circles. The endpoints of the original line segment and the point of intersection of the two circles are the vertices of an equilateral triangle. Each side has a length of two units and the measure of each angle is 60°. From geometry, the altitude drawn from one of the vertices bisects the angle at that vertex and also bisects the opposite side to which it is perpendicular. Two right triangles are formed that have acute angles of 30° and 60°, a hypotenuse of two units, and a short leg of one unit. Using Pythagoras' theorem, the long leg is $\sqrt{3}$ units. The trigonometric functions of 30° and 60° are then calculated as follows:

$$\sin 60° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}$$

$$\cos 60° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \tan 30° = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \text{ rationalizing the denominator}$$

The geometric derivation of the values of the sine, cosine and tangent functions for the 'special' acute angles 30°, 45° and 60°, in Examples 1 and 2, agree with the results from the previous chapter. The results for these angles – in both degree and radian measure – are summarised in the box below.

Values of sine, cosine and tangent for common acute angles

$\sin 30° = \sin\frac{\pi}{6} = \frac{1}{2}$ $\quad \cos 30° = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ $\quad \tan 30° = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$

$\sin 45° = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad \cos 45° = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\quad \tan 45° = \tan\frac{\pi}{4} = 1$

$\sin 60° = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ $\quad \cos 60° = \cos\frac{\pi}{3} = \frac{1}{2}$ $\quad \tan 60° = \tan\frac{\pi}{3} = \sqrt{3}$

● **Hint:** It is important that you are able to recall – without a calculator – the exact trigonometric values for these common angles.

Observe that $\sin 30° = \cos 60° = \frac{1}{2}$, $\sin 60° = \cos 30° = \frac{\sqrt{3}}{2}$ and $\sin 45° = \cos 45° = \frac{\sqrt{2}}{2}$. Complementary angles (sum of 90°) have equal function values for sine and cosine. That is, for all angles x measured in degrees, $\sin x = \cos(90° - x)$ or $\sin(90° - x) = \cos x$. As noted in Chapter 6, it is for this reason that sine and cosine are called co-functions.

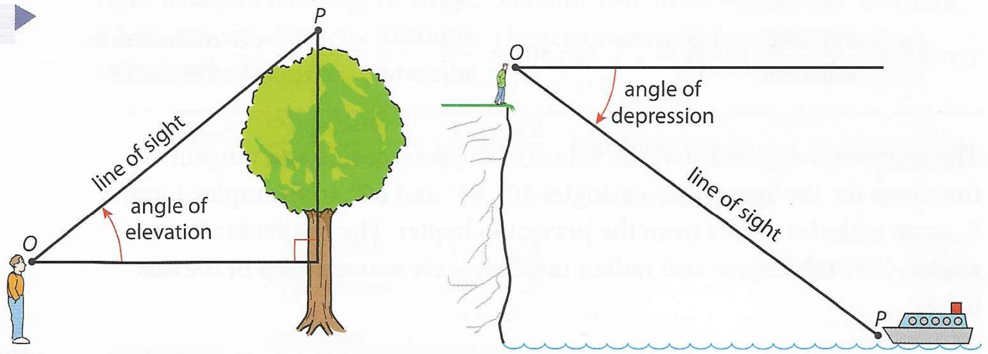
Solution of right triangles

Every triangle has three sides and three angles – six different parts. The ancient Greeks knew how to solve for all of the unknown angles and sides in a right triangle given that either the length of two sides, or the length of one side and the measure of one angle, were known. To **solve a right triangle** means to find the measure of any unknown sides or angles. We can accomplish this by applying Pythagoras' theorem and trigonometric functions. We will utilize trigonometric functions in two different ways when solving for missing parts in right triangles – to find the length of a side, and to find the measure of an angle. Solving right triangles using the sine, cosine and tangent functions is essential to finding solutions to problems in fields such as astronomy, navigation, engineering and architecture. In Sections 7.3 and 7.4, we will see how trigonometry can also be used to solve for missing parts in triangles that are not right triangles.

Angles of depression and elevation

An imaginary line segment from an observation point O to a point P (representing the location of an object) is called the **line of sight** of P. If P is above O, the acute angle between the line of sight of P and a horizontal line passing through O is called the **angle of elevation** of P. If P is below O, the angle between the line of sight and the horizontal is called the **angle of depression** of P. This is illustrated in Figure 7.5.

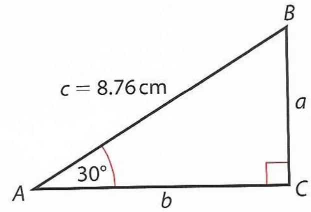
Figure 7.5



Example 3

Solve triangle ABC given $c = 8.76$ cm and angle $A = 30°$, where the right angle is at C. Give exact answers when possible, otherwise give to an accuracy of 3 significant figures.

Solution



Knowing that the conventional notation is to use a lower-case letter to represent the length of a side opposite the vertex denoted with the corresponding upper-case letter, we sketch triangle ABC indicating the known measurements.

From the definition of sine and cosine functions, we have

$$\sin 30° = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{a}{8.76} \qquad \cos 30° = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{b}{8.76}$$

$$a = 8.76 \sin 30° \qquad b = 8.76 \cos 30°$$

$$a = 8.76\left(\frac{1}{2}\right) = 4.38 \qquad b = 8.76\left(\frac{\sqrt{3}}{2}\right) \approx 7.586\,382\,537 \approx 7.59$$

Therefore, $a = 4.38$ cm, $b \approx 7.59$ cm, and it's clear that angle $B = 60°$.

We can use Pythagoras' theorem to check our results for a and b.

$$a^2 + b^2 = c^2 \Rightarrow \sqrt{a^2 + b^2} = 8.76$$

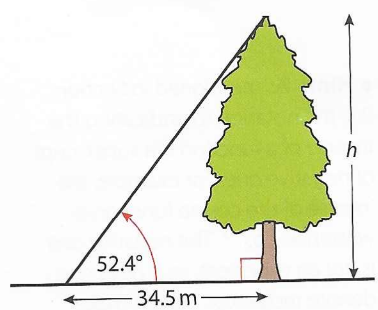
Be aware that the result for a is exactly 4.38 cm (assuming measurements given for angle A and side c are exact), but the result for b can only be approximated. To reduce error when performing the check, we should use the most accurate value (i.e. most significant figures) for b possible. The most effective way to do this on our GDC is to use results that are stored to several significant figures, as shown in the GDC screen image.

```
8.76(√(3)/2)
          7.586382537
Ans→B
          7.586382537
√(4.38²+B²)
                 8.76
```

Example 4

A scientist involved in forest management wants to measure the height of a tree without climbing it. From a point 34.5 m from the base of a large tree, the scientist determines that the angle of elevation from horizontal ground to the top of the tree is 52.4°. What is the height of the tree, approximated to the nearest tenth of a metre?

Solution



$$\tan 52.4° = \frac{\text{opposite}}{\text{adjacent}} = \frac{h}{34.5} \Rightarrow h = 34.5 \tan 52.4°$$

$$h \approx 34.5(1.2985)$$

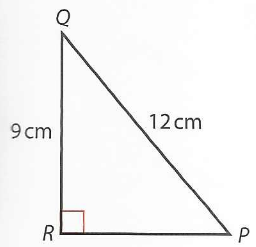
$$h \approx 44.799\,16$$

The height of the tree is approximately 44.8 m.

In both Examples 3 and 4, one of the acute angles of a right triangle was given so the third angle is easily determined from the fact that the sum of the angles is 180°. Let's look at how we can use trigonometric functions to solve a right triangle for which the lengths of two of the sides are known, but the measure of both acute angles are unknown.

Example 5

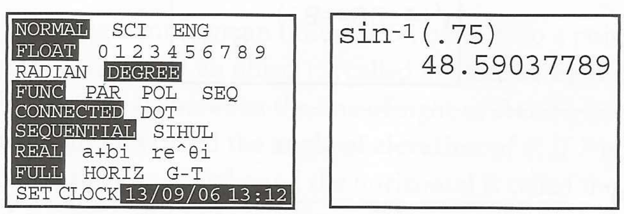
Solve triangle PQR given $QR = 9$ cm and $PQ = 12$ cm, where the right angle is at R. Give exact answers when possible, otherwise give to an accuracy of 3 s.f.



Solution

Using Pythagoras' theorem: $PR = \sqrt{12^2 - 9^2} = \sqrt{63} = 3\sqrt{7} \approx 7.94$.

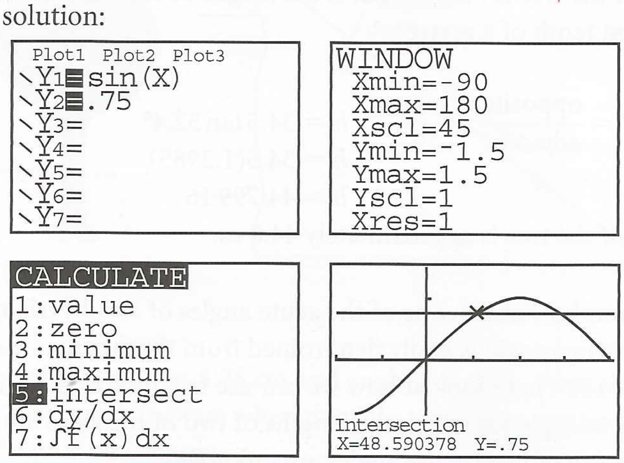
Both of the acute angles, $\angle P$ and $\angle Q$, are unknown. We know the lengths of all three sides, so it is possible to evaluate any of the trigonometric functions for either of these angles. For example, it is clear that $\sin P = \frac{9}{12} = 0.75$. To determine the acute angle that has a sine ratio of 0.75, we need to perform the **inverse** of the sine function (written as $\sin^{-1}$). We can do this by solving the equation $\sin x = 0.75$ graphically, as we did in Section 6.4. (See GDC screen images at bottom of page.) Using a graphical method is particularly suitable if x represents a real number, or perhaps an angle in radian measure, and there is more than one solution for x. For triangle PQR, there is only one solution for P in the equation $\sin P = 0.75$ and it must be between 0° and 90°. Your GDC (in 'degree' mode) can be used to find the acute angle P, either graphically or by directly computing the inverse sine of 0.75. Although, as we will realize, there are an infinite number of angles with a sine ratio of 0.75, your GDC is programmed so that the inverse sine ($\sin^{-1}$) computation gives only the one acute angle with a sine ratio of 0.75. The GDC screen images illustrate that having your GDC compute an inverse trigonometric value is the most efficient method for finding an acute angle.



Thus, $\angle P \approx 48.6°$ from which it follows that $\angle Q \approx 90° - 48.6° \approx 41.4°$.

Therefore, the missing parts of triangle PQR are $PR \approx 7.94$ cm, $\angle P \approx 48.6°$ and $\angle Q \approx 41.4°$.

Graphical solution:

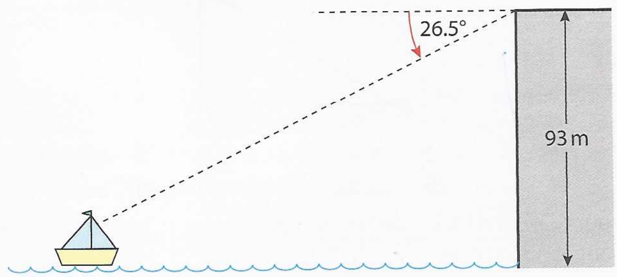


• **Hint:** As mentioned in Section 2.3, the notation for indicating the inverse of a function is a superscript of negative one. For example, the inverse of the cosine function is written as $\cos^{-1}$. The negative one is *not* an exponent, so it does not denote reciprocal. Do not make this error: $\cos^{-1} x \neq \frac{1}{\cos x}$. And as stated in Section 2.3, if $f(a) = b$ then $f^{-1}(b) = a$. For example, for the sine function, if $\sin 60° = \frac{\sqrt{3}}{2}$ then $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60°$.

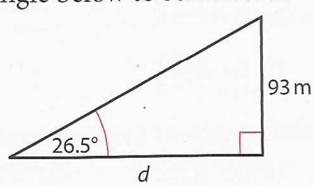
Example 6

From the top of a perpendicular cliff 93 m high, the angle of depression of a boat is 26.5°. How far is the boat from the foot of the cliff? Give your answer accurate to 3 s.f.

Solution



If the angle of depression of the boat from the top of the cliff is 26.5°, the angle of elevation of the top of the cliff from the boat is also 26.5°. Thus, we can use the right triangle below to solve for d.



$$\tan 26.5° = \frac{93}{d} \Rightarrow d = \frac{93}{\tan 26.5} \Rightarrow d \approx \frac{93}{0.498\,58} \approx 186.53$$

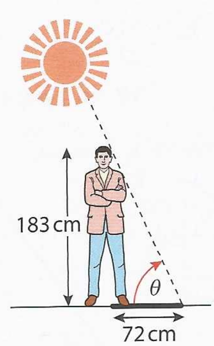
The boat is approximately 187 m from the foot of the cliff.

Example 7

A man who is 183 cm tall casts a 72 cm long shadow on the horizontal ground. What is the angle of elevation of the sun to the nearest tenth of a degree?

Solution

In the diagram, the angle of elevation of the sun is labelled θ.



$$\tan \theta = \frac{183}{72}$$

$$\theta = \tan^{-1}\left(\frac{183}{72}\right)$$

$$\theta \approx 68.5°$$

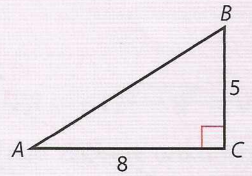
```
tan-1(183/72)
         68.52320902
```

GDC computation

The angle of elevation of the sun is approximately 68.5°.

Exercise 7.1

In questions 1–6, find the exact value of the trigonometric function for the specified acute angle in triangle ABC.



1 $\sin A$ **2** $\cos A$ **3** $\tan A$

4 $\sin B$ **5** $\cos B$ **6** $\tan B$

7 Using your GDC, find (accurate to 3 s.f.) the degree measure of $B\hat{A}C$ and $A\hat{B}C$ in right triangle ABC above.

In questions 8–13, sketch a right triangle corresponding to the given trigonometric function of the acute angle θ. Use Pythagoras' theorem to determine the third side, and then find the value of the other two trigonometric functions of θ.

8 $\sin\theta = \frac{3}{5}$ **9** $\cos\theta = \frac{5}{8}$ **10** $\tan\theta = 2$

11 $\cos\theta = \frac{7}{10}$ **12** $\tan\theta = \frac{1}{3}$ **13** $\cos\theta = \frac{\sqrt{7}}{4}$

In questions 14–19, find the exact value of the trigonometric function.

14 $\sin 45°$ **15** $\cos\frac{\pi}{6}$ **16** $\tan 45°$

17 $\sin\frac{\pi}{3}$ **18** $\tan\frac{\pi}{6}$ **19** $\cos 60°$

In questions 20–25, find the exact value of θ in degree measure ($0 < \theta < 90°$) and in radian measure $\left(0 < \theta < \frac{\pi}{2}\right)$ without using your GDC.

20 $\cos\theta = \frac{1}{2}$ **21** $\sin\theta = \frac{\sqrt{2}}{2}$ **22** $\tan\theta = \sqrt{3}$

23 $\sin\theta = \frac{\sqrt{3}}{2}$ **24** $\tan\theta = 1$ **25** $\cos\theta = \frac{\sqrt{3}}{2}$

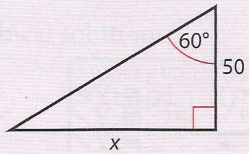
In questions 26–31, find the approximate value (to 3 s.f.) of θ in degree measure ($0 < \theta < 90°$) and in radian measure $\left(0 < \theta < \frac{\pi}{2}\right)$ by using the inverse key on your GDC.

26 $\sin\theta = 0.7258$ **27** $\cos\theta = 0.7258$ **28** $\tan\theta = 1.2953$

29 $\cos\theta = 0.1638$ **30** $\sin\theta = 0.4721$ **31** $\tan\theta = 0.6507$

In questions 32–37, solve for x. Give your answer to 3 s.f.

32



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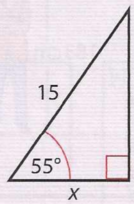

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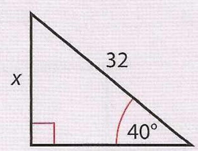

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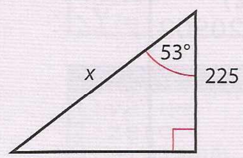

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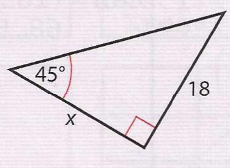

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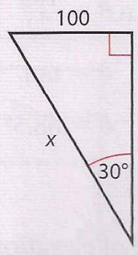
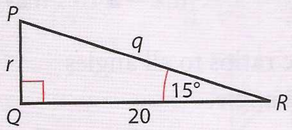


In questions 38 and 39, solve for all of the unknown sides and angles.

38



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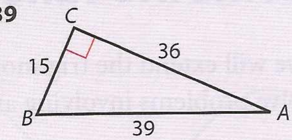
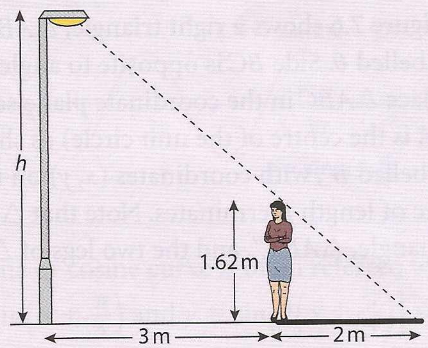


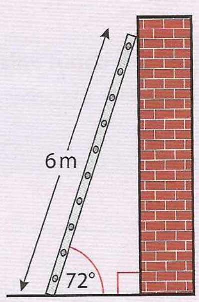
40 The tallest tree in the world is reputed to be a giant redwood named *Hyperion* located in Redwood National Park in California, USA. At a point 41.5 m from the centre of its base and on the same elevation, the angle of elevation of the top of the tree is 70°. How tall is the tree? Give your answer to 3 s.f.

41 The top of the Eiffel Tower in Paris (not including the antenna) is 300 m high. What will be the angle of elevation of the top of the tower from a point on the ground (assumed level) that is 125 m from the centre of the tower's base?

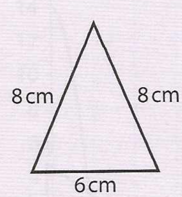
42 A woman, 1.62 m tall, standing 3 m from a street light casts a 2 m long shadow (see diagram). What is the height of the street light?



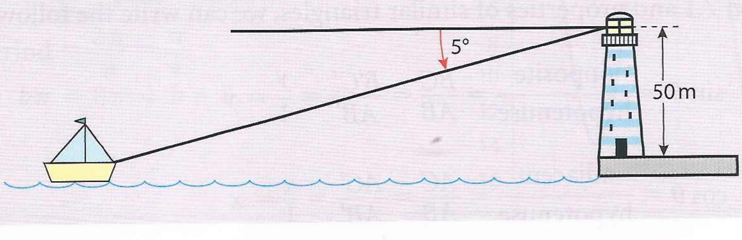
43 A 6 m ladder leaning against the side of a building makes a 72° angle with the ground (see diagram). How far up the side of the house does the ladder reach?



44 An isosceles triangle has sides of length 8 cm, 8 cm and 6 cm (see diagram). Find the angle between the two equal sides.



45 From a 50 m observation tower on the shoreline, a coastguard sights a boat in difficulty. The angle of depression of the boat is 5° (see diagram). How far is the boat from the shoreline?



7.2 Trigonometric functions of any angle

In this section, we will extend the trigonometric ratios to all angles allowing us to solve problems involving any size angle.

Functions of an angle related to functions of a real number

It is useful to pause for a moment in our consideration of the trigonometric functions as functions of an acute angle in a right triangle, and take a look at how this approach relates to the one taken in Chapter 6, where the trigonometric functions were functions of a real number.

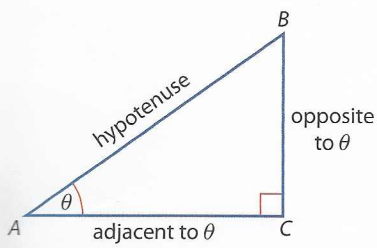


Figure 7.6

Figure 7.6 shows a right triangle, $\triangle ABC$, where the angle at vertex A is labelled θ. Side BC is opposite to angle θ and side AC is adjacent to angle θ. Place $\triangle ABC$ in the coordinate plane so that angle θ is in standard position (A is the centre of the unit circle) as shown in Figure 7.7. The point labelled B', with coordinates (x, y) on the unit circle, is the point where the arc of length t terminates. Note that $\triangle ABC$ is similar to the smaller right triangle, $\triangle AB'C'$, and the two legs of $\triangle AB'C'$ are x and y (Figure 7.8).

Figure 7.7 The radian measure of angle θ is t.

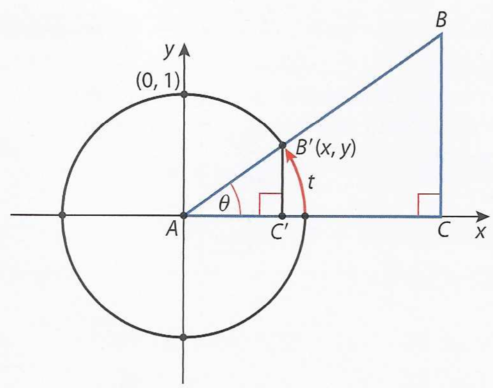
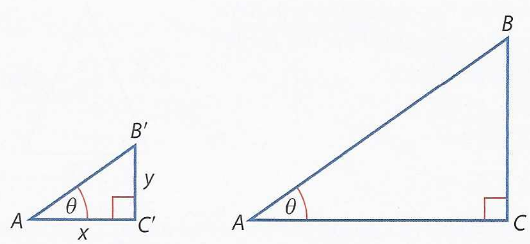


Figure 7.8



From the definitions of the trigonometric functions of an acute angle in Section 7.1 and properties of similar triangles, we can write the following:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AB} = \frac{B'C'}{AB'} = \frac{y}{1} = y$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AC}{AB} = \frac{AC'}{AB'} = \frac{x}{1} = x$$

From the definitions of the trigonometric functions for the real number t in Section 6.2, we know that $\sin t = y$ and $\cos t = x$. Furthermore, if θ is given in radian measure, then $\theta = t$. Therefore, the trigonometric functions **of the angle** with radian measure θ are precisely the same as the trigonometric functions **of the real number** t. One of the reasons why trigonometric functions are so useful in a range of applications is because they can be applied in these two different ways.

Now let's consider angles other than acute angles.

Defining trigonometric functions for any angle in standard position

Consider the point $P(x, y)$ on the terminal side of an angle θ in standard position (Figure 7.9) such that r is the distance from the origin O to P. If θ is an acute angle then we can construct a right triangle POQ (Figure 7.10) by dropping a perpendicular from P to a point Q on the x-axis, and it follows that:

$$\sin \theta = \frac{y}{r}, \cos \theta = \frac{x}{r} \text{ and } \tan \theta = \frac{y}{x}\ (x \neq 0).$$

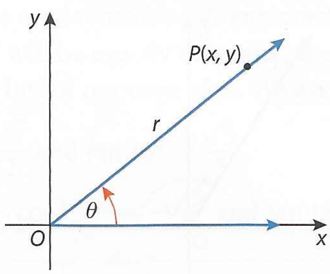


Figure 7.9

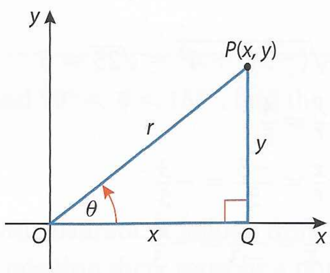


Figure 7.10

Extending this to angles other than acute angles allows us to define the trigonometric functions for any angle – positive or negative. It is important to note that the values of the trigonometric ratios do not depend on the choice of the point $P(x, y)$. If $P'(x', y')$ is any other point on the terminal side of angle θ, as in Figure 7.11, then triangles POQ and $P'OQ'$ are similar and the trigonometric ratios for corresponding angles are equal.

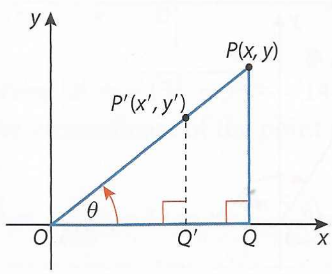
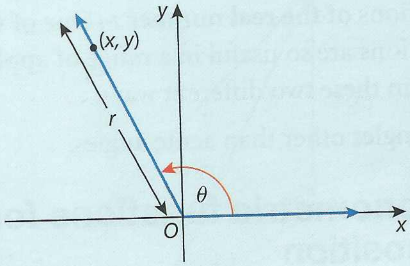


Figure 7.11

Definition of trigonometric functions

Let θ be any angle (in degree or radian measure) in standard position, with (x, y) any point on the terminal side of θ, and $r = \sqrt{x^2 + y^2}$, the distance from the origin to the point (x, y), as shown below.

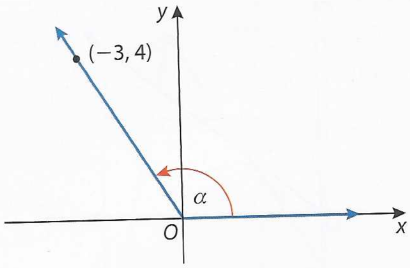


Then the trigonometric functions are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}\ (x \neq 0)$$

Example 8

Find the sine, cosine and tangent of an angle α that contains the point $(-3, 4)$ on its terminal side when in standard position.



Solution

$$r = \sqrt{x^2 + y^2} = \sqrt{(-3)^2 + 4^2} = \sqrt{25} = 5$$

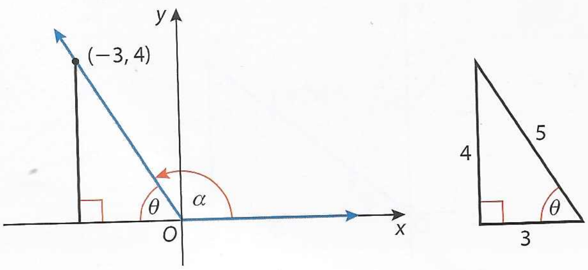
Then, $\sin\alpha = \frac{y}{r} = \frac{4}{5}$

$$\cos\alpha = \frac{x}{r} = \frac{-3}{5} = -\frac{3}{5}$$

$$\tan\alpha = \frac{y}{x} = \frac{4}{-3} = -\frac{4}{3}$$

Note that for the angle α in Example 8, we can form a right triangle by constructing a line segment from the point $(-3, 4)$ perpendicular to the x-axis, as shown in Figure 7.12. Clearly, $\theta = 180° - \alpha$. Furthermore, the values of the sine, cosine and tangent of the angle θ are the same as that for the angle α, except that the *sign* may be different.

Figure 7.12



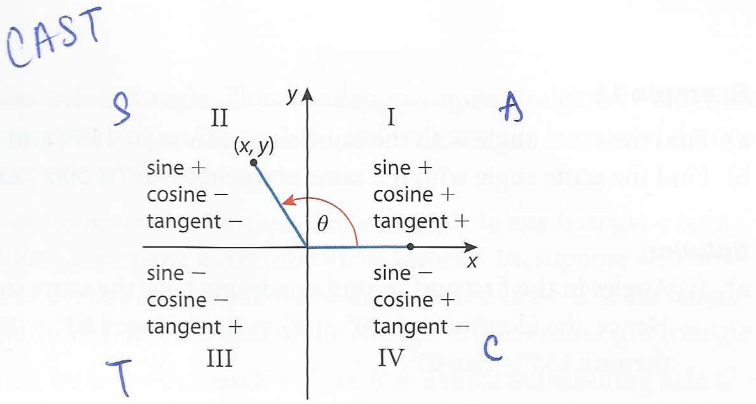


Figure 7.13 Sign of trigonometric function values depends on the quadrant in which the terminal side of the angle lies.

Whether the trigonometric functions are defined in terms of the length of an arc or in terms of an angle, the signs of trigonometric function values are determined by the quadrant in which the arc or angle lies, when in standard position (Figure 7.13).

Example 9

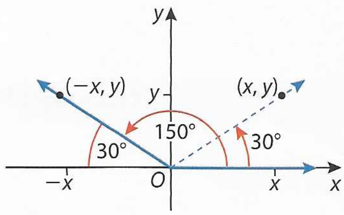
Find the sine, cosine and tangent of the obtuse angle that measures 150°.

Solution

The terminal side of the angle forms a 30° angle with the x-axis. The sine values for 150° and 30° will be exactly the same, and the cosine and tangent values will be the same but of opposite sign. We know that

$\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$ and $\tan 30° = \frac{\sqrt{3}}{3}$.

Therefore, $\sin 150° = \frac{1}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}$ and $\tan 150° = -\frac{\sqrt{3}}{3}$.



Example 9 illustrates three trigonometric identities for angles whose sum is 180° (i.e. a pair of supplementary angles). The following are true for any acute angle θ:

$\sin(180° - \theta) = \sin\theta$

$\cos(180° - \theta) = -\cos\theta$

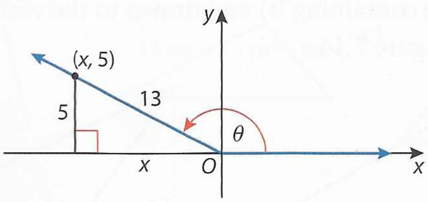
$\tan(180° - \theta) = -\tan\theta$

Example 10

Given that $\sin\theta = \frac{5}{13}$ and $90° < \theta < 180°$, find the exact values of $\cos\theta$ and $\tan\theta$.

Solution

θ is an angle in the second quadrant. It follows from the definition $\sin\theta = \frac{y}{r}$ that with θ in standard position there must be a point on the terminal side of the angle that is 13 units from the origin (i.e. $r = 13$) and which has a y-coordinate of 5, as shown in the diagram.



Using Pythagoras' theorem, $|x| = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$. Because θ is in the second quadrant, the x-coordinate of the point must be negative, thus $x = -12$.

Therefore, $\cos\theta = \frac{-12}{13} = -\frac{12}{13}$, and $\tan\theta = \frac{5}{-12} = -\frac{5}{12}$.

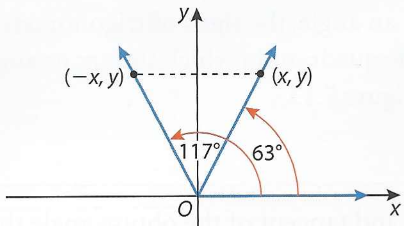
Example 11

a) Find the acute angle with the same sine ratio as (i) 135°, and (ii) 117°.
b) Find the acute angle with the same cosine ratio as (i) 300°, and (ii) 342°.

Solution

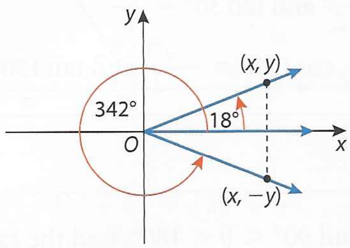
a) (i) Angles in the first and second quadrants have the same sine ratio. Hence, the identity $\sin(180° - \theta) = \sin\theta$. Since $180° - 135° = 45°$, then $\sin 135° = \sin 45°$.

(ii) Since $180° - 117° = 63°$, then $\sin 117° = \sin 63°$



b) (i) Angles in the first and fourth quadrants have the same cosine ratio. Hence, the identity $\cos(360° - \theta) = \cos\theta$. Since $360° - 300° = 60°$, then $\cos 300° = \cos 60°$.

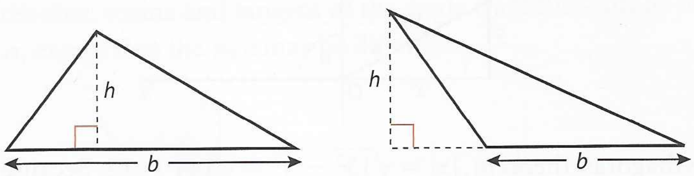
(ii) Since $360° - 342° = 18°$, then $\cos 342° = \cos 18°$.



Areas of triangles

You are familiar with the standard formula for the area of a triangle, area $= \frac{1}{2} \times$ base $\times$ height (or area $= \frac{1}{2}bh$), where the base, b, is a side of the triangle and the height, h, (or altitude) is a line segment perpendicular to the base (or the line containing it) and drawn to the vertex opposite to the base, as shown in Figure 7.14.

Figure 7.14



If the lengths of two sides of a triangle and the measure of the angle between these sides (often called the included angle) are known, then the triangle is unique and has a fixed area. Hence, we should be able to calculate the area from just these measurements, i.e. from knowing two sides and the included angle. This calculation is quite straightforward if the triangle is a right triangle (Figure 7.15) and we know the lengths of the two legs on either side of the right angle.

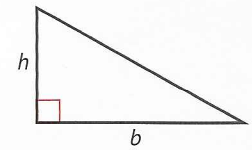


Figure 7.15

Let's develop a general area formula that will apply to any triangle – right, acute or obtuse. For triangle ABC shown in Figure 7.16, suppose we know the lengths of the two sides a and b and the included angle C. If the length of the height from B is h, the area of the triangle is $\frac{1}{2}bh$. From right triangle trigonometry, we know that $\sin C = \frac{h}{a}$, or $h = a\sin C$. Substituting $a\sin C$ for h, area $= \frac{1}{2}bh = \frac{1}{2}b(a\sin C) = \frac{1}{2}ab\sin C$.

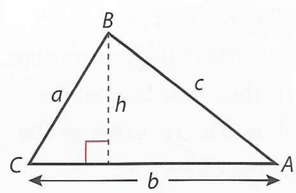


Figure 7.16

If the angle C is obtuse, then from Figure 7.17 we see that $\sin(180° - C) = \frac{h}{a}$. So, the height is $h = a\sin(180° - C)$. However, $\sin(180° - C) = \sin C$. Thus, $h = a\sin C$ and, again, area $= \frac{1}{2}ab\sin C$.

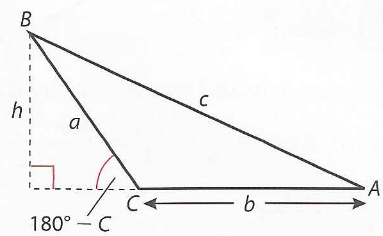


Figure 7.17

Area of a triangle

For a triangle with sides of lengths a and b and included angle C,

$$\text{Area of } \triangle = \frac{1}{2}ab\sin C$$

● **Hint:** Note that the procedure for finding the area of a triangle from a pair of sides and the included angle can be performed three different ways. For any triangle labelled in the manner of the triangles in Figures 7.16 and 7.17, its area is expressed by any of the following three expressions.

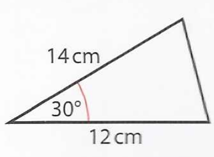
$$\text{Area of } \triangle = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B = \frac{1}{2}bc\sin A$$

These three equivalent expressions will prove to be helpful for developing an important formula for solving non-right triangles in the next section.

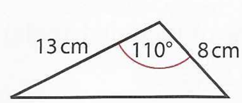
Example 12

Find the area of each triangle. Express the area exactly, or, if not possible, express it accurate to 3 s.f.

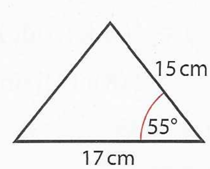
a)



b)



c)



Solution

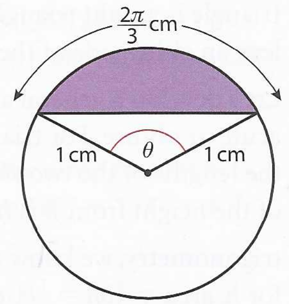
a) Area $= \frac{1}{2}(12)(14)\sin 30° = 84(0.5) = 42\text{ cm}^2$

b) Area $= \frac{1}{2}(8)(13)\sin 110° \approx 52(0.939\,69) \approx 48.9\text{ cm}^2$

c) Area $= \frac{1}{2}(15)(17)\sin 55° \approx 127.5(0.819\,152) \approx 104\text{ cm}^2$

Example 13

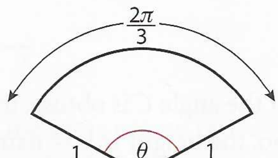
The circle shown has a radius of 1 cm and the central angle θ subtends an arc of length of $\frac{2\pi}{3}$ cm. Find the area of the shaded region.

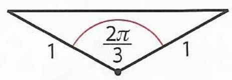


Solution

The formula for the area of a sector is $A = \frac{1}{2}r^2\theta$ (Section 6.1), where θ is the central angle in radian measure. Since the radius of the circle is one, the length of the arc subtended by θ is the same as the radian measure of θ. Thus, area of sector $= \frac{1}{2}(1)^2\left(\frac{2\pi}{3}\right) = \frac{\pi}{3}$ cm^2. The area of the triangle formed by the two radii and the chord is equal to $\frac{1}{2}(1)(1)\sin\left(\frac{2\pi}{3}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$ cm^2.

$\left[\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{2\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\right]$





The area of the shaded region is found by subtracting the area of the triangle from the area of the sector. Area $= \frac{\pi}{3} - \frac{\sqrt{3}}{4}$ or $\frac{4\pi - 3\sqrt{3}}{12}$ or approximately 0.614 cm^2 (3 s.f.).

Example 14

Show that it is possible to construct two different triangles with an area of 35 cm^2 that have sides measuring 8 cm and 13 cm. For each triangle, find the measure of the (included) angle between the sides of 8 cm and 13 cm to the nearest tenth of a degree.

Solution

We can visualize the two different triangles with equal areas – one with an acute included angle (α) and the other with an obtuse included angle (β).

$$\text{Area} = \tfrac{1}{2}(\text{side})(\text{side})(\text{sine of included angle}) = 35 \text{ cm}^2$$

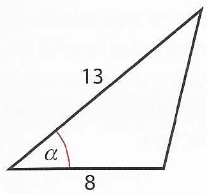
$$= \tfrac{1}{2}(8)(13)(\sin\alpha) = 35$$

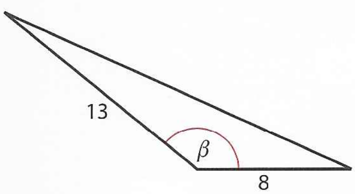
$$52\sin\alpha = 35$$

$$\sin\alpha = \frac{35}{52}$$

$\alpha = \sin^{-1}\left(\frac{35}{52}\right)$ recall that the GDC will only give the acute angle with sine ratio of $\frac{35}{52}$

$\alpha \approx 42.3°$ rounded to the nearest tenth





Knowing that $\sin(180° - \alpha) = \sin\alpha$, the obtuse angle β is equal to $180° - 42.3° = 137.7°$.

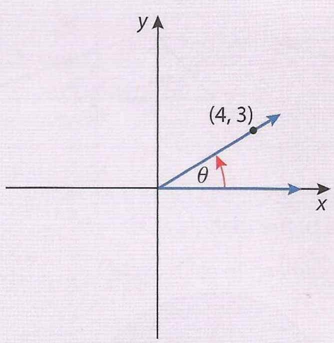
Check this answer by computing on your GDC:
$\frac{1}{2}(8)(13)(\sin 137.7°) \approx 34.997 \approx 35\text{ cm}^2$.

Therefore, there are two different triangles with sides 8 cm and 13 cm and area of 35 cm^2 – one with an included angle of 42.3° and the other with an included angle of 137.7°.

Exercise 7.2

In questions 1–6, find the exact value of the sine, cosine and tangent functions of the angle θ.

1



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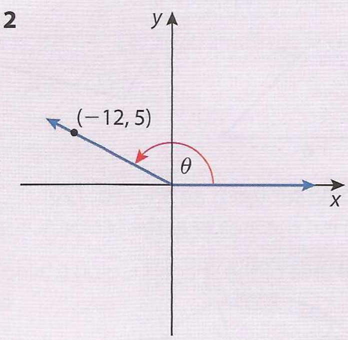

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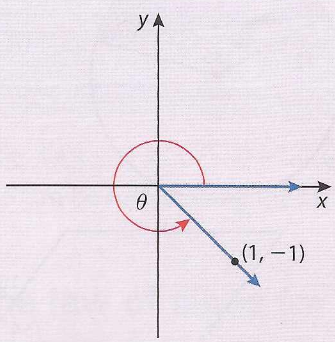

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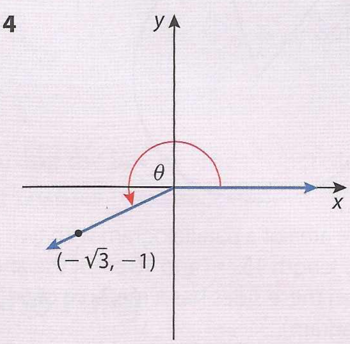

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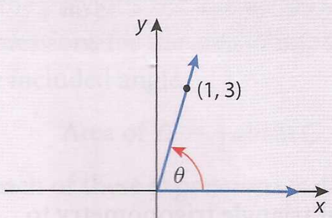

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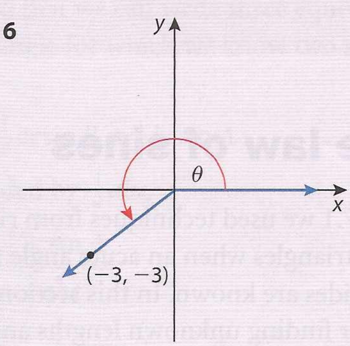


7 By using the symmetry of the unit circle, or otherwise, determine the exact sine, cosine and tangent function values for the following common obtuse angles.

a) 120° b) 135° c) 150°

8 Evaluate the sine, cosine and tangent of each angle without using your GDC.

a) 225° b) 330° c) $\frac{7\pi}{6}$

d) −60° e) 270° f) $\frac{5\pi}{3}$

g) −120° h) $-\frac{\pi}{4}$ i) π

9 Given that $\cos\theta = \frac{3}{5}$ and $-90° < \theta < 0°$, find the exact values of $\sin\theta$ and $\tan\theta$.

10 Given that $\sin\theta = \frac{8}{17}$ and $90° < \theta < 180°$, find the exact values of $\cos\theta$ and $\tan\theta$.

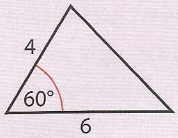
11 Given that $\tan\theta = -\frac{12}{5}$ and $\sin\theta < 0°$, find the exact values of $\sin\theta$ and $\cos\theta$.

12 Given that $\sin\theta = 0$ and $\cos\theta < 0°$, find the exact values of $\cos\theta$ and $\tan\theta$.

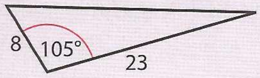
13 a) Find the acute angle with the same sine ratio as (i) 150°, and (ii) 95°.
b) Find the acute angle with the same cosine ratio as (i) 315°, and (ii) 353°.
c) Find the acute angle with the same tangent ratio as (i) 240°, and (ii) 200°.

14 Find the area of each triangle. Express the area exactly, or, if not possible, express it accurate to 3 s.f.

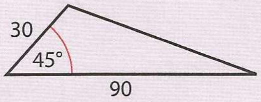
a)



b)



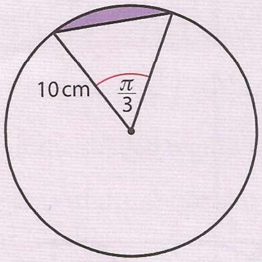
c)



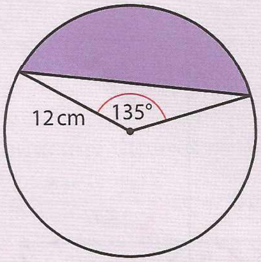
15 A chord *AB* subtends an angle of 120° at *O*, the centre of a circle with radius 15 cm. Find the area of a) the sector *AOB*, and b) the triangle *AOB*.

16 Find the area of the shaded region (called a *segment*) in each circle.

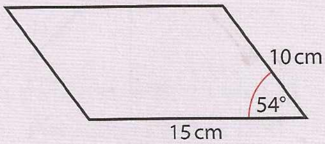
a)



b)



17 Find the area of a parallelogram with two sides of length 15 cm and 10 cm, if the angle between these sides has a measure of 54° (see diagram).



7.3 The law of sines

In Section 7.1 we used techniques from right triangle trigonometry to solve right triangles when an acute angle and one side are known, or when two sides are known. In this section and the next, we will study methods for finding unknown lengths and angles in triangles that are not right triangles. These general methods are effective for solving problems involving any kind of triangle – right, acute or obtuse.

Possible triangles constructed from three given parts

As mentioned in the previous paragraph, we've solved right triangles by either knowing an acute angle and one side, or knowing two sides. Since the triangles also have a right angle, each of those two cases actually