involved knowing three different parts of the triangle – either two angles and a side, or two sides and an angle. We need to know at least three parts of a triangle in order to solve for other unknown parts. Different arrangements of the three known parts can be given. Before solving for unknown parts, it is helpful to know whether the three known parts determine a unique triangle, or possibly more than one triangle. The table below summarizes the five different arrangements of three parts and the number of possible triangles for each. You are encouraged to confirm these results on your own with manual or computer generated sketches.

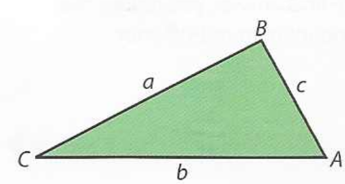
Possible triangles formed with three known parts

Known parts	Number of possible triangles
Three angles (AAA)	Infinite triangles (not possible to solve)
Three sides (SSS) (sum of any two must be greater than the third)	One unique triangle
Two sides and their included angle (SAS)	One unique triangle
Two angles and any side (ASA or AAS)	One unique triangle
Two sides and a non-included angle (SSA)	No triangle, one triangle or two triangles

ASA, AAS and SSA can be solved using the **law of sines**, whereas SSS and SAS can be solved using the **law of cosines** (next section).

The law of sines (or sine rule)

In the previous section, we showed that we can write three equivalent expressions for the area of any triangle for which we know two sides and the included angle.



$$\text{Area of } \triangle = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}bc\sin A$$

If each of these expressions is divided by $\frac{1}{2}abc$,

$$\frac{\frac{1}{2}ab\sin C}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac\sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}bc\sin A}{\frac{1}{2}abc}$$

we obtain three equivalent ratios – each containing the sine of an angle divided by the length of the side opposite the angle.

The law of sines

If A, B and C are the angle measures of any triangle and a, b and c are, respectively, the lengths of the sides opposite these angles, then

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

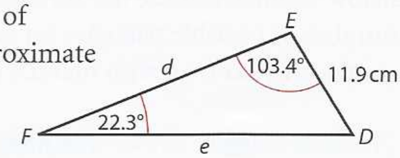
Alternatively, the law of sines can also be written as $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

Solving triangles given two angles and any side (ASA or AAS)

If we know two angles and any side of a triangle, we can use the law of sines to find any of the other angles or sides of the triangle.

Example 15

Find all of the unknown angles and sides of triangle *DEF* shown in the diagram. Approximate all measurements to 1 decimal place.



Solution

The third angle of the triangle is

$$D = 180° - E - F = 180° - 103.4° - 22.3° = 54.3°$$

Using the law of sines, we can write the following proportion to solve for the length *e*:

$$\frac{\sin 22.3°}{11.9} = \frac{\sin 103.4°}{e}$$

$$e = \frac{11.9 \sin 103.4°}{\sin 22.3°} \approx 30.507 \text{ cm}$$

We can write another proportion from the law of sines to solve for *d*:

$$\frac{\sin 22.3°}{11.9} = \frac{\sin 54.3°}{d}$$

$$d = \frac{11.9 \sin 54.3°}{\sin 22.3°} \approx 25.467 \text{ cm}$$

Therefore, the other parts of the triangle are $D = 54.3°$, $e \approx 30.5$ cm and $d \approx 25.5$ cm.

• **Hint:** When using your GDC to find angles and lengths with the law of sines (or the law of cosines), remember to store intermediate answers on the GDC for greater accuracy. By not rounding until the final answer, you reduce the amount of round-off error.

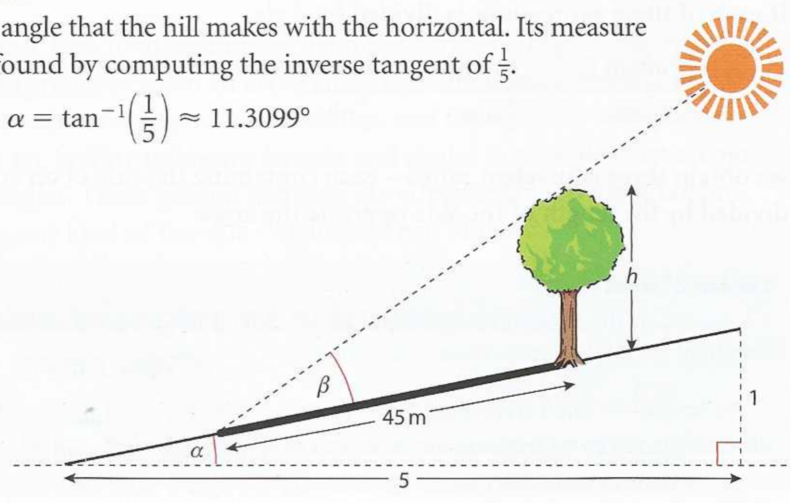
Example 16

A tree on a sloping hill casts a shadow 45 m along the side of the hill. The gradient of the hill is $\frac{1}{5}$ (or 20%) and the angle of elevation of the sun is 35°. How tall is the tree to the nearest tenth of a metre?

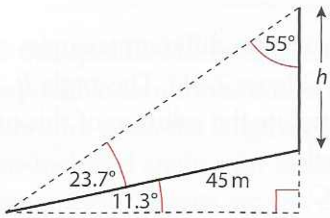
Solution

α is the angle that the hill makes with the horizontal. Its measure can be found by computing the inverse tangent of $\frac{1}{5}$.

$$\alpha = \tan^{-1}\left(\frac{1}{5}\right) \approx 11.3099°$$



The height of the tree is labelled h. The angle of elevation of the sun is the angle between the sun's rays and the horizontal. In the diagram, this angle of elevation is the sum of α and β. Thus, $\beta \approx 35° - 11.3099° \approx 23.6901°$. For the larger right triangle with $\alpha + \beta = 35°$ as one of its acute angles, the other acute angle – and the angle in the obtuse triangle opposite the side of 45 m – must be 55°. Now we can apply the law of sines for the obtuse triangle to solve for h.



$$\frac{\sin 23.7°}{h} = \frac{\sin 55°}{45} \Rightarrow h = \frac{45 \sin 23.7°}{\sin 55°} \approx 22.0809$$

Therefore, the tree is approximately 22.1 m tall.

Two sides and a non-included angle (SSA) – the ambiguous case

The arrangement where we are given the lengths of two sides of a triangle and the measure of an angle not between those two sides can produce three different results: no triangle, one unique triangle or two different triangles. Let's explore these possibilities with the following example.

Example 17

Find all of the unknown angles and sides of triangle ABC where $a = 35$ cm, $b = 50$ cm and $A = 30°$. Approximate all measurements to 1 decimal place.

Solution

Figure 7.18 shows the three parts we have from which to try and construct a triangle.

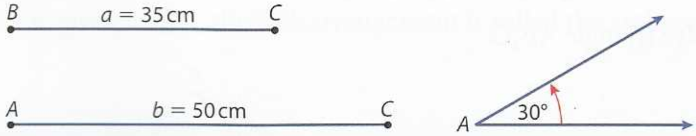
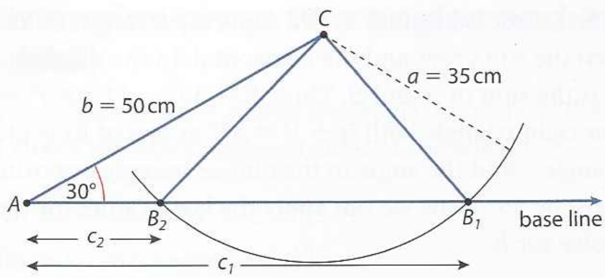


Figure 7.18

We attempt to construct the triangle, as shown in Figure 7.19. We first draw angle A with its initial side (or base line of the triangle) extended. We then measure off the known side $b = AC = 50$. To construct side a (opposite angle A), we take point C as the centre and with radius $a = 35$ we draw an arc of a circle. The points on this arc are all possible positions for vertex B – one of the endpoints of side a, or BC. Point B must be on the base line, so B can be located at any point of intersection of the circular arc and the base line. In this instance, with these particular measurements for the two sides and non-included angle, there are two points of intersection, which we label B_1 and B_2.

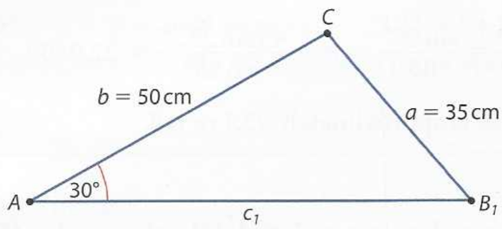
Figure 7.19



Therefore, we can construct two different triangles, triangle AB_1C (Figure 7.20) and triangle AB_2C (Figure 7.21). The angle B_1 will be acute and angle B_2 will be obtuse. To complete the solution of this problem, we need to solve each of these triangles.

- Solve triangle AB_1C:

Figure 7.20



We can solve for acute angle B_1 using the law of sines:

$$\frac{\sin 30°}{35} = \frac{\sin B_1}{50}$$

$$\sin B_1 = \frac{50 \sin 30°}{35} = \frac{50(0.5)}{35}$$

$$\mathrm{B}_1 = \sin^{-1}\left(\frac{5}{7}\right) \approx 45.5847°$$

Then, $C \approx 180° - 30° - 45.5847° \approx 104.4153°$.

With another application of the law of sines, we can solve for side c_1:

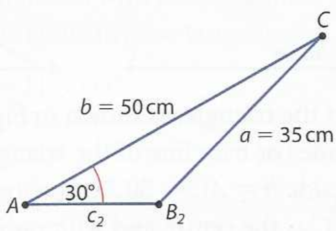
$$\frac{\sin 30°}{35} = \frac{\sin 104.4153°}{c_1}$$

$$c_1 = \frac{35 \sin 104.4153°}{\sin 30°} \approx \frac{35(0.96852)}{0.5} \approx 67.7964 \text{ cm}$$

Therefore, for triangle AB_1C, $B_1 \approx 45.6°$, $C \approx 104.4°$ and $c_1 \approx 67.8$ cm.

- Solve triangle AB_2C:

Figure 7.21



Solving for obtuse angle B_2 using the law of sines gives the same result as above, except we know that $90° < B_2 < 180°$.

We also know that $\sin(180° - \theta) = \sin \theta$.

Thus, $B_2 = 180° - B_1 \approx 180° - 45.5847° \approx 134.4153°$.

Then, $C \approx 180° - 30° - 134.4153° \approx 15.5847°$.

With another application of the law of sines, we can solve for side c_2:

$$\frac{\sin 30°}{35} = \frac{\sin 15.5847°}{c_2}$$

$$c_2 \approx \frac{35 \sin 15.5847°}{\sin 30°} \approx \frac{35(0.26866)}{0.5} \approx 18.8062 \text{ cm}$$

Therefore, for triangle AB_2C, $B_2 \approx 134.4°$, $C \approx 15.6°$ and $c_2 \approx 18.8$ cm.

Now that we have solved this specific example, let's take a more general look and examine all the possible conditions and outcomes for the SSA arrangement. In general, we are given the lengths of two sides – call them a and b – and a non-included angle – for example, angle A that is opposite side a. From these measurements, we can determine the number of different triangles. Figure 7.22 shows the four different possibilities (or cases) when angle A is acute. The number of triangles depends on the length of side a.

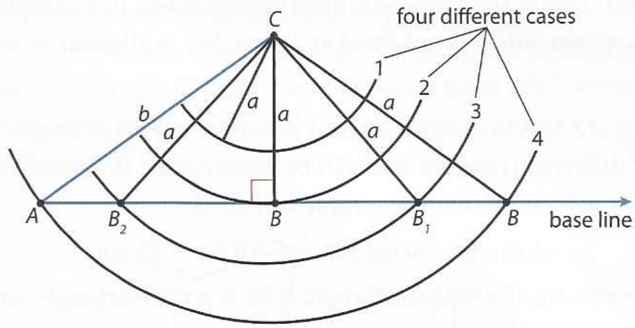


Figure 7.22

In case 2, side a is perpendicular to the base line resulting in a single right triangle, shown in Figure 7.23. In this case, clearly $\sin A = \frac{a}{b}$ and $a = b\sin A$. In case 1, the length of a is shorter than it is in case 2, i.e $b\sin A$. In case 3, which occurred in Example 17, the length of a is longer than $b\sin A$, but less than b. And, in case 4, the length of a is greater than b. These results are summarized in the table below. Because the number of triangles may be none, one or two, depending on the length of a (the side opposite the given angle), the SSA arrangement is called the ambiguous case.

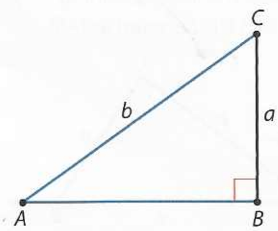


Figure 7.23

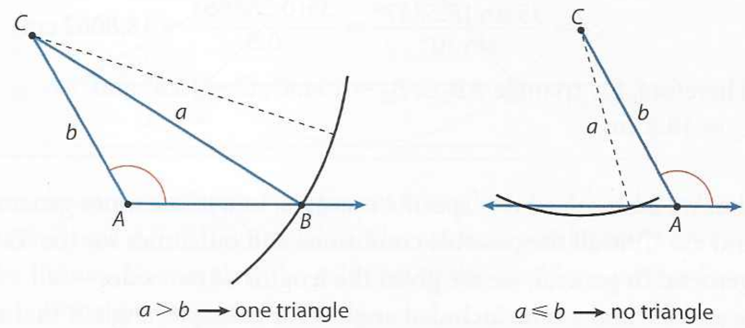
The ambiguous case (SSA)

Given the lengths of sides a and b and the fact that the non-included angle A is acute, the following four cases and resulting triangles can occur.

Length of a	Number of triangles
$a < b\sin A$	No triangle
$a = b\sin A$	One right triangle
$b\sin A < a < b$	Two triangles
$a \geqslant b$	One triangle

The situation is considerably simpler if angle A is obtuse rather than acute. Figure 7.24 shows that if $a > b$ then there is only one possible triangle, and if $a \leqslant b$ then no triangle that contains angle A is possible.

Figure 7.24 Angle A is obtuse.



Example 18

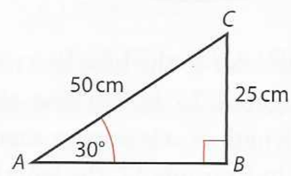
For triangle ABC, if side $b = 50$ cm and angle $A = 30°$, find the values for the length of side a that will produce: (i) no triangle, (ii) one triangle, (iii) two triangles. This is the same SSA information given in Example 17 with the exception that side a is not fixed at 35 cm, but is allowed to vary.

Solution

Because this is a SSA arrangement and given A is an acute angle, then the number of different triangles that can be constructed is dependent on the length of a. First calculate the value of $b\sin A$:

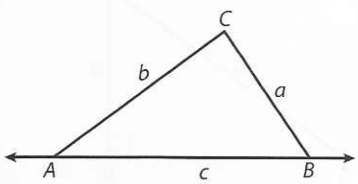
$$b\sin A = 50\sin 30° = 50(0.5) = 25 \text{ cm}$$

Thus, if a is exactly 25 cm then triangle ABC is a right triangle, as shown in the figure.



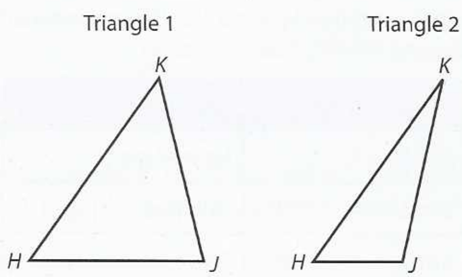
(i) If $a < 25$ cm, there is no triangle.
(ii) If $a = 25$ cm, or $a > 50$ cm, there is one unique triangle.
(iii) If $25 \text{ cm} < a < 50$ cm, there are two different possible triangles.

• **Hint:** It is important to be familiar with the notation for line segments and angles commonly used in IB exam questions. For example, the line segment labelled b in the diagram (below) is denoted as $[AC]$ in IB notation. Angle A, the angle between $[BA]$ and $[AC]$, is denoted as $B\hat{A}C$. Also, the line containing points A and B is denoted as (AB).



Example 19

The diagrams below show two different triangles both satisfying the conditions: $HK = 18$ cm, $JK = 15$ cm, $J\hat{H}K = 53°$.



a) Calculate the size of $H\hat{J}K$ in Triangle 2.
b) Calculate the area of Triangle 1.

Solution

a) From the law of sines, $\frac{\sin(H\hat{J}K)}{18} = \frac{\sin 53°}{15} \Rightarrow \sin(H\hat{J}K) = \frac{18 \sin 53°}{15}$

$\approx 0.958\,36 \Rightarrow \sin^{-1}(0.95836) \approx 73.408°$

However, $H\hat{J}K > 90° \Rightarrow H\hat{J}K \approx 180° - 73.408° \approx 106.592°$.

Therefore, in Triangle 2 $H\hat{J}K \approx 107°$ (3 s.f.).

b) In Triangle 1, $H\hat{J}K < 90° \Rightarrow H\hat{J}K \approx 73.408°$

$\Rightarrow H\hat{K}J \approx 180° - (73.408° + 53°) \approx 53.592°$

Area $= \frac{1}{2}(18)(15)\sin(53.592°) \approx 108.649\text{ cm}^2$.

Therefore, the area of Triangle 1 is approximately 109 cm^2 (3 s.f.).

7.4 The law of cosines

Two cases remain in our list of different ways to arrange three known parts of a triangle. If three sides of a triangle are known (SSS arrangement), or two sides of a triangle and the angle between them are known (SAS arrangement), then a unique triangle is determined. However, in both of these cases, the law of sines cannot solve the triangle.

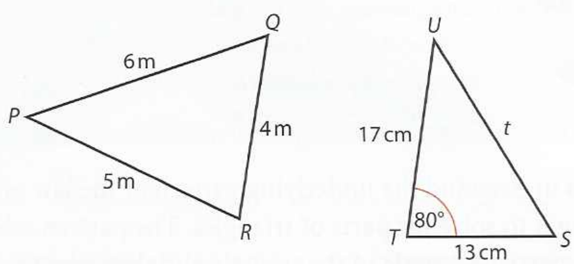


Figure 7.25

For example, it is not possible to set up an equation using the law of sines to solve triangle PQR or triangle STU in Figure 7.25.

- Trying to solve $\triangle PQR$: $\frac{\sin P}{4} = \frac{\sin R}{6} \Rightarrow$ two unknowns; cannot solve for angle P or angle R
- Trying to solve $\triangle STU$: $\frac{\sin 80°}{t} = \frac{\sin U}{13} \Rightarrow$ two unknowns; cannot solve for angle U or side R

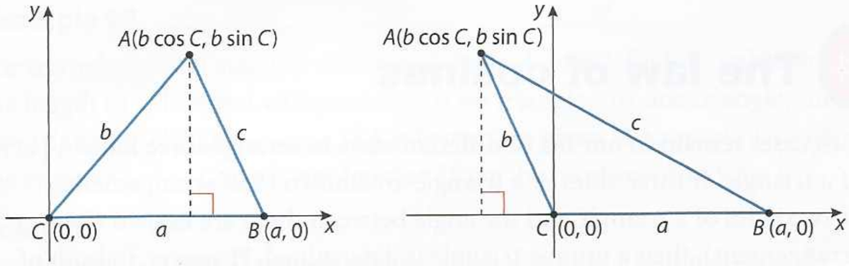
The law of cosines (or cosine rule)

We will need the **law of cosines** to solve triangles with these kinds of arrangements of sides and angles. To derive this law, we need to place a general triangle ABC in the coordinate plane so that one of the vertices is at the origin and one of the sides is on the positive x-axis. Figure 7.26 shows both an acute triangle ABC and an obtuse triangle ABC. In either case, the coordinates of vertex C are $x = b\cos C$ and $y = b\sin C$. Because c is the distance from A to B, then we can use the distance formula to write

$c = \sqrt{(b\cos C - a)^2 + (b\sin C - 0)^2}$	distance between $(b\cos C, b\sin C)$ and $(a, 0)$
$c^2 = (b\cos C - a)^2 + (b\sin C - 0)^2$	squaring both sides
$c^2 = b^2\cos^2 C - 2ab\cos C + a^2 + b^2\sin^2 C$	expand
$c^2 = b^2(\cos^2 C + \sin^2 C) - 2ab\cos C + a^2$	factor out b^2 from two terms
$c^2 = b^2 - 2ab\cos C + a^2$	apply trigonometric identity $\cos^2\theta + \sin^2\theta = 1$
$c^2 = a^2 + b^2 - 2ab\cos C$	rearrange terms

This equation gives one form of the law of cosines. Two other forms are obtained in a similar manner by having either vertex A or vertex B, rather than C, located at the origin.

Figure 7.26

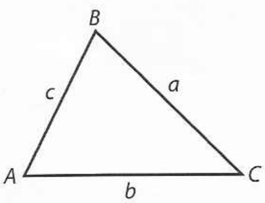


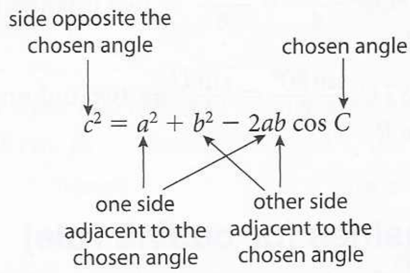
The law of cosines

In any triangle ABC with corresponding sides a, b and c:

$$c^2 = a^2 + b^2 - 2ab\cos C$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

It is helpful to understand the underlying pattern of the law of cosines when applying it to solve for parts of triangles. The pattern relies on choosing one particular angle of the triangle and then identifying the two sides that are adjacent to the angle and the one side that is opposite to it. The law of cosines can be used to solve for the chosen angle or the side opposite the chosen angle.



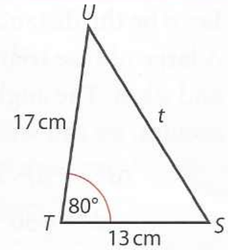


Solving triangles given two sides and the included angle (SAS)

If we know two sides and the included angle, we can use the law of cosines to solve for the side opposite the given angle. Then it is best to solve for one of the two remaining angles using the law of sines.

Example 20

Find all of the unknown angles and sides of triangle STU, one of the triangles shown earlier in Figure 7.25. Approximate all measurements to 1 decimal place.



Solution

We first solve for side t, opposite the known angle $S\hat{T}U$, using the law of cosines:

$$t^2 = 13^2 + 17^2 - 2(13)(17)\cos 80°$$

$$t = \sqrt{13^2 + 17^2 - 2(13)(17)\cos 80°}$$

$$t \approx 19.5256$$

Now use the law of sines to solve for one of the other angles, say $T\hat{S}U$:

$$\frac{\sin T\hat{S}U}{17} = \frac{\sin 80°}{19.5256}$$

$$\sin T\hat{S}U = \frac{17\sin 80°}{19.5256}$$

$$T\hat{S}U = \sin^{-1}\left(\frac{17\sin 80°}{19.5256}\right)$$

$$T\hat{S}U \approx 59.0288°$$

Then, $S\hat{U}T \approx 180° - (80° + 59.0288°) \approx 40.9712°$.

Therefore, the other parts of the triangle are $t \approx 19.5$ cm, $T\hat{S}U \approx 59.0°$ and $S\hat{U}T \approx 41.0°$.

• **Hint:** As previously mentioned, remember to store intermediate answers on the GDC for greater accuracy. By not rounding until the final answer, you reduce the amount of round-off error. The GDC screen images below show the calculations in the solution for Example 20 above.

```
√(13²+17-2(13)(
17)cos(80))
        19.52556031
Ans→T
        19.52556031
```

```
Ans→T
        19.52556031
sin⁻¹(17sin(80)/T
)
        59.02884098
Ans→S
        59.02884098
```

```
sin⁻¹(17sin(80)/T
)
        59.02884098
Ans→S
        59.02884098
180-(80+S)
        40.97115902
```

You may have noticed that the formula for the law of cosines looks similar to the formula for Pythagoras' theorem. In fact, Pythagoras' theorem can be considered a special case of the law of cosines. When the chosen angle in the law of cosines is 90°, and since $\cos 90° = 0$, the law of cosines becomes Pythagoras' theorem.

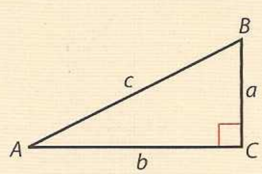
If angle $C = 90°$, then

$c^2 = a^2 + b^2 - 2ab\cos C$

$\Rightarrow c^2 = a^2 + b^2 - 2ab\cos 90°$

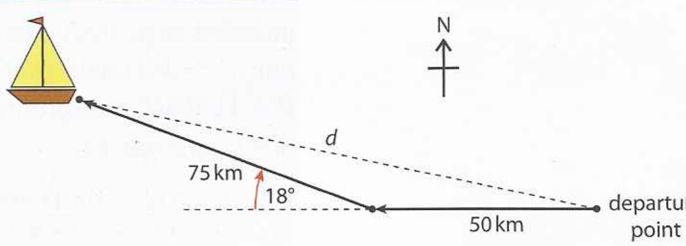
$\Rightarrow c^2 = a^2 + b^2 - 2ab(0)$

$\Rightarrow c^2 = a^2 + b^2$ or $a^2 + b^2 = c^2$



Example 21

A ship travels 50 km due west, then changes its course 18° northward, as shown in the diagram. After travelling 75 km in that direction, how far is the ship from its point of departure? Give your answer to the nearest tenth of a kilometre.



Solution

Let d be the distance from the departure point to the position of the ship. A large obtuse triangle is formed by the three distances of 50 km, 75 km and d km. The angle opposite side d is $180° - 18° = 162°$. Using the law of cosines, we can write the following equation to solve for d:

$$d^2 = 50^2 + 75^2 - 2(50)(75)\cos 162°$$

$$d = \sqrt{50^2 + 75^2 - 2(50)(75)\cos 162°} \approx 123.523$$

Therefore, the ship is approximately 123.5 km from its departure point.

Solving triangles given three sides (SSS)

Given three line segments such that the sum of the lengths of any two is greater than the length of the third, then they will form a unique triangle. Therefore, if we know three sides of a triangle we can solve for the three angle measures. To use the law of cosines to solve for an unknown angle, it is best to first rearrange the formula so that the chosen angle is the subject of the formula.

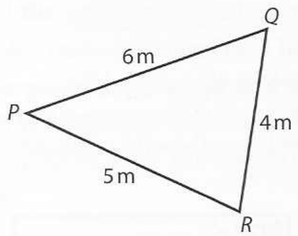
Solve for angle C in:

$$c^2 = a^2 + b^2 - 2ab\cos C \Rightarrow 2ab\cos C = a^2 + b^2 - c^2 \Rightarrow \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Then, $C = \cos^{-1}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$.

Example 22

Find all of the unknown angles of triangle PQR, the second triangle shown earlier in Figure 7.25. Approximate all measurements to 1 decimal place.



Solution

Note that the smallest angle will be opposite the shortest side. Let's first solve for the smallest angle – thus, writing the law of cosines with chosen angle P:

$$P = \cos^{-1}\left(\frac{5^2 + 6^2 - 4^2}{2(5)(6)}\right) \approx 41.4096°$$

Now that we know the measure of angle P, we have two sides and a non-included angle (SSA), and the law of sines can be used to find the other non-included angle. Consider the sides $QR = 4$, $RP = 5$ and the angle $P \approx 41.4096°$. Substituting into the law of sines, we can solve for angle Q that is opposite RP.

$$\frac{\sin Q}{5} = \frac{\sin 41.4096°}{4}$$

$$\sin Q = \frac{5\sin 41.4096°}{4}$$

$$Q = \sin^{-1}\left(\frac{5\sin 41.4096°}{4}\right) \approx 55.7711°$$

Then, $R \approx 180° - (41.4096° + 55.7711°) \approx 82.8192°$.

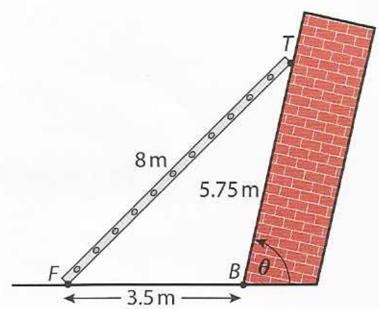
Therefore, the three angles of triangle PQR are $P \approx 41.4°$, $Q \approx 55.8°$ and $R \approx 82.8°$.

Example 23

A ladder that is 8 m long is leaning against a non-vertical wall that slopes away from the ladder. The foot of the ladder is 3.5 m from the base of the wall, and the distance from the top of the ladder down the wall to the ground is 5.75 m. To the nearest tenth of a degree, what is the acute angle at which the wall is inclined to the horizontal?

Solution

Let's start by drawing a diagram that accurately represents the given information. θ marks the acute angle of inclination of the wall. Its supplement is $F\hat{B}T$. From the law of cosines:



$$\cos F\hat{B}T = \frac{3.5^2 + 5.75^2 - 8^2}{2(3.5)(5.75)}$$

$$F\hat{B}T = \cos^{-1}\left(\frac{3.5^2 + 5.75^2 - 8^2}{2(3.5)(5.75)}\right) \approx 117.664°$$

$$\theta \approx 180° - 117.664° \approx 62.336°$$

Therefore, the angle of inclination of the wall is approximately 62.3°.

Exercise 7.3 and 7.4

In questions1–6, state the number of distinct triangles (none, one, two or infinite) that can be constructed with the given measurements. If the answer is one or two triangles, provide a sketch of each triangle.

1 $A\hat{C}B = 30°$, $A\hat{B}C = 50°$ and $B\hat{A}C = 100°$

2 $A\hat{C}B = 30°$, $AC = 12$ cm and $BC = 17$ cm

3 $A\hat{C}B = 30°$, $AB = 7$ cm and $AC = 14$ cm

4 $A\hat{C}B = 47°$, $BC = 20$ cm and $A\hat{B}C = 55°$

5 $B\hat{A}C = 25°$, $AB = 12$ cm and $BC = 7$ cm

6 $AB = 23$ cm, $AC = 19$ cm and $BC = 11$ cm

In questions 7–15, solve the triangle. In other words, find the measurements of all unknown sides and angles. If two triangles are possible, solve for both.

7 $B\hat{A}C = 37°$, $A\hat{B}C = 28°$ and $AC = 14$

8 $A\hat{B}C = 68°$, $A\hat{C}B = 47°$ and $AC = 23$

9 $B\hat{A}C = 18°$, $A\hat{C}B = 51°$ and $AC = 4.7$

10 $A\hat{C}B = 112°$, $A\hat{B}C = 25°$ and $BC = 240$

11 $BC = 68$, $A\hat{C}B = 71°$ and $AC = 59$

12 $BC = 16$, $AC = 14$ and $AB = 12$

13 $BC = 42$, $AC = 37$ and $AB = 26$

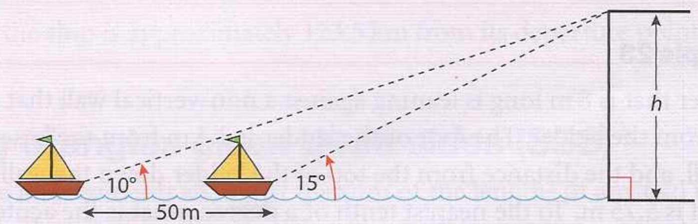
14 $BC = 34$, $A\hat{B}C = 43°$ and $AC = 28$

15 $AC = 0.55$, $B\hat{A}C = 62°$ and $BC = 0.51$

16 Find the lengths of the diagonals of a parallelogram whose sides measure 14 cm and 18 cm and which has one angle of 37°.

17 Find the measures of the angles of an isosceles triangle whose sides are 10 cm, 8 cm and 8 cm.

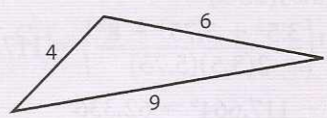
18 A boat is sailing directly towards a cliff. The angle of elevation of a point on the top of the cliff and straight ahead of the boat increases from 10° to 15° as the ship sails a distance of 50 m (see diagram). Find the height of the cliff.



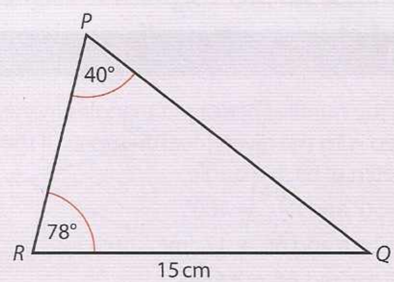
19 Given that for triangle DEF, $E\hat{D}F = 43°$, $DF = 24$ and $FE = 18$, find the two possible measures of $D\hat{F}E$.

20 A tractor drove from a point A directly north for 500 m, and then drove north-east (i.e. bearing of 45°) for 300 m, stopping at point B. What is the distance between points A and B?

21 Find the measure of the smallest angle in the triangle shown.



22 Find the area of triangle PQR.

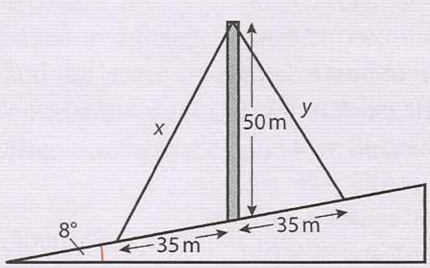


In questions 23 and 24, find a value for the length of BC so that the number of possible triangles is: a) one, b) two and c) none.

23 $B\hat{A}C = 36°$, $AB = 5$

24 $B\hat{A}C = 60°$, $AB = 10$

25 A 50 m vertical pole is to be erected on the side of a sloping hill that makes a 8° angle with the horizontal (see diagram). Find the length of each of the two supporting wires (x and y) that will be anchored 35 m uphill and downhill from the base of the pole.



7.5 Applications

There are some additional applications of triangle trigonometry – both right triangles and non-right triangles – that we should take some time to examine.

Equations of lines and angles between two lines

Recall from Section 1.6, the slope m, or gradient, of a non-vertical line is defined as $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{vertical change}}{\text{horizontal change}}$.

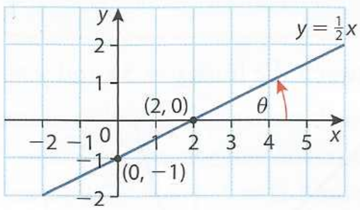


Figure 7.27

The equation of the line shown in Figure 7.27 has a slope $m = \frac{1}{2}$ and a y-intercept of $(0, -1)$. So, the equation of the line is $y = \frac{1}{2}x - 1$. We can find the measure of the acute angle θ between the line and the x-axis by using the tangent function (Figure 7.28).

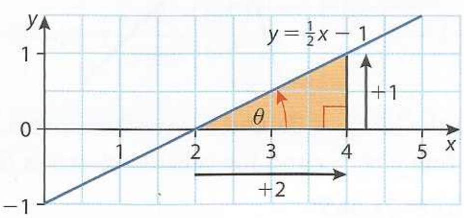


Figure 7.28

$\theta = \tan^{-1}(m) = \tan^{-1}\left(\frac{1}{2}\right) \approx 26.6°$.

Clearly, the slope, m, of this line is equal to $\tan\theta$. If we know the angle between the line and the x-axis, and the y-intercept $(0, c)$, we can write the equation of the line in slope-intercept form ($y = mx + c$) as $y = (\tan\theta)x + c$.

Before we can generalize for any non-horizontal line, let's look at a line with a negative slope.

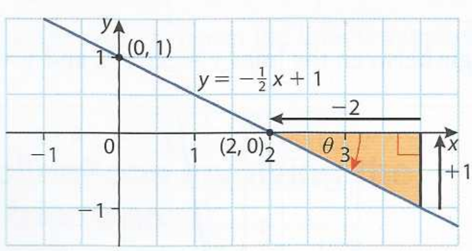


Figure 7.29

The slope of the line is $-\frac{1}{2}$. In order for $\tan\theta$ to be equal to the slope of the line, the angle θ must be the angle that the line makes with the x-axis in the positive direction, as shown in Figure 7.29. In this example,
$\theta = \tan^{-1}(m) = \tan^{-1}\left(-\frac{1}{2}\right) \approx -26.6°$.

Remember, an angle with a negative measure indicates a clockwise rotation from the initial side to the terminal side of the angle.

Equations of lines intersecting the x-axis

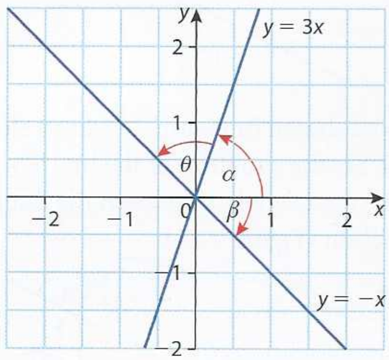
If a line has a y-intercept of $(0, c)$ and makes an angle of θ with the positive direction of the x-axis, such that $-90° < \theta < 90°$, then the slope (gradient) of the line is $m = \tan\theta$ and the equation of the line is $y = (\tan\theta)x + c$. Note: The angle this line makes with any horizontal line will be θ.

Let's use triangle trigonometry to find the angle between any two intersecting lines – not just for a line intersecting the x-axis. Realize that any pair of intersecting lines that are not perpendicular will have both an acute angle and an obtuse angle between them. When asked for an angle between two lines, the convention is to give the acute angle.

Example 24

Find the acute angle between the lines $y = 3x$ and $y = -x$.

Solution



The angle between the line $y = 3x$ and the positive x-axis is α, and the angle between the line $y = -x$ and the positive x-axis is β.

$$\alpha = \tan^{-1}(3) \approx 71.565°$$
$$\beta = \tan^{-1}(-1) = -45°$$

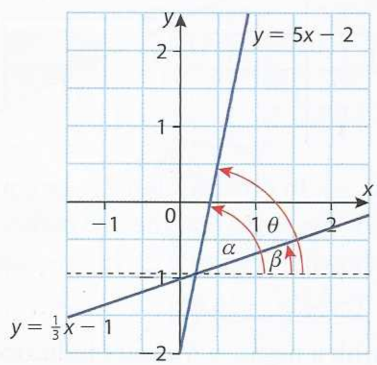
The obtuse angle between the two lines is
$\alpha - \beta \approx 71.565° - (-45°) \approx 116.565°$.

Therefore, the acute angle θ between the two lines is
$\theta = 180° - 116.565° \approx 63.4°$.

Example 25

Find the acute angle between the lines $y = 5x - 2$ and $y = \frac{1}{3}x - 1$.

Solution



A horizontal line is drawn through the point of intersection.

The angle between $y = 5x - 2$ and this horizontal line is α, and the angle between $y = \frac{1}{3}x - 1$ and this horizontal line is β.

$$\alpha = \tan^{-1}(5) \approx 78.690° \quad \text{and} \quad \beta = \tan^{-1}\left(\tfrac{1}{3}\right) = 18.435°$$

The acute angle θ between the two lines is $\theta = \alpha - \beta \approx 78.690° - 18.435° \approx 60.3°$.

We can generalize the procedure for finding the angle between two lines as follows.

Given two non-vertical lines with equations of $y_1 = m_1x + c_1$ and $y_2 = m_2x + c_2$, the angle between the two lines is $|\tan^{-1}(m_1) - \tan^{-1}(m_2)|$. Note: This angle may be acute or obtuse.

Example 26

a) Find the exact equation of line L_1 that passes through the origin and makes an angle of $-60°$ with the positive direction of the x-axis (or 120°).

b) The equation of line L_2 is $7x + y + 1 = 0$. Find the acute angle between the lines L_1 and L_2.

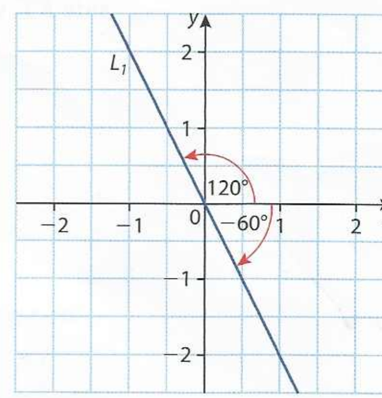
Solution

a) The equation of the line is given by $y = (\tan\theta)x$

$$\Rightarrow y = [\tan(-60°)]x = \left[\frac{\sin(-60°)}{\cos(-60°)}\right]x = \left[\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}}\right]x = (-\sqrt{3})x$$

Therefore, the equation of L_1 is $y = (-\sqrt{3})x$ or $y = -x\sqrt{3}$.

Note: $\tan(-60°) = \tan 120° = -\sqrt{3}$.



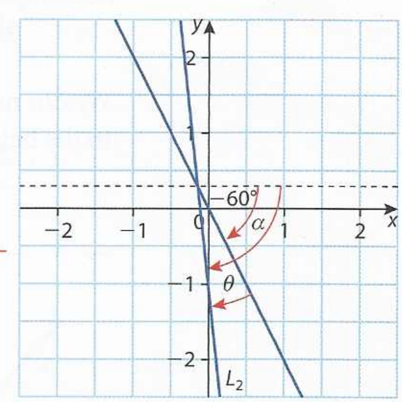
b) L_2: $7x + y + 1 = 0 \Rightarrow y = -7x - 1$

θ is the acute angle between the lines L_1 and L_2.

$\theta = |\tan^{-1}(m_1) - \tan^{-1}(m_2)| = |\tan^{-1}(-\sqrt{3}) - \tan^{-1}(-7)|$

$\Rightarrow \theta \approx |-60° - (-81.870°)| \approx |-21.87°|$

Therefore, the acute angle between the lines is approximately 21.9° (3 s.f.).

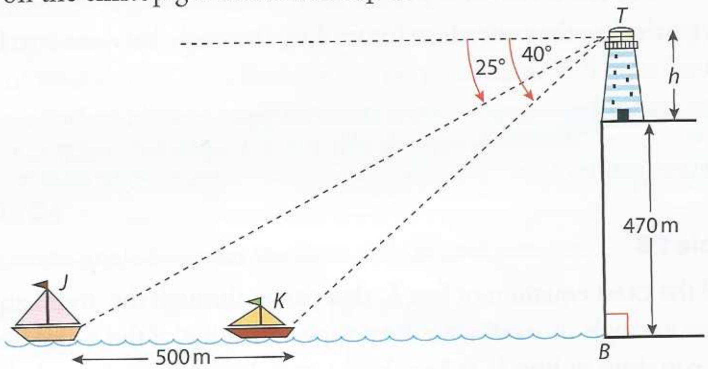


Further applications involving the solution of triangles

Many problems that involve distances and angles are represented by diagrams with multiple triangles – right and otherwise. These diagrams can be confusing and difficult to interpret correctly. In these situations, it is important to carry out a careful analysis of the given information and diagram – this will usually lead to drawing additional diagrams. Often we can extract a triangle, or triangles, for which we have enough information to allow us to solve the triangle(s).

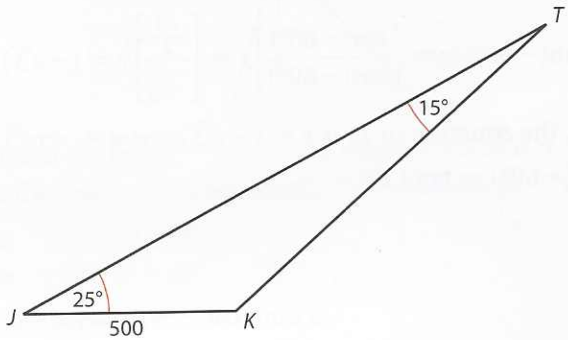
Example 27

Two boats, J and K, are 500 m apart. A lighthouse is on top of a 470 m cliff. The base, B, of the cliff is in line horizontally with $[JK]$. From the top, T, of the lighthouse, the angles of depression of J and K are, respectively, 25° and 40°. Find, correct to the nearest metre, the height, h, of the lighthouse from its base on the clifftop ground to the top T.



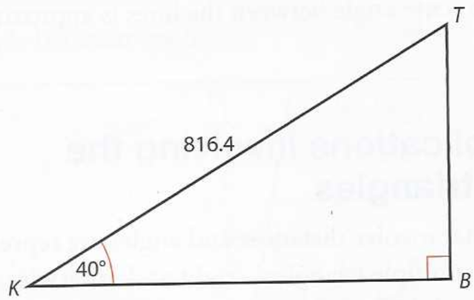
Solution

First, extract obtuse triangle JKT and apply the law of sines to solve for the side KT, which is also the hypotenuse of the right triangle KBT.



$$\frac{\sin 25°}{KT} = \frac{\sin 15°}{500} \Rightarrow KT = \frac{500 \sin 25°}{\sin 15°} \approx 816.436 \text{ m}$$

We can now use the right triangle KBT to find the side BT – which is equal to the height of the cliff plus the height of the lighthouse.



$$\sin 40° = \frac{BT}{816.436} \Rightarrow BT = 816.436 \sin 40° \approx 524.795 \text{ m}$$

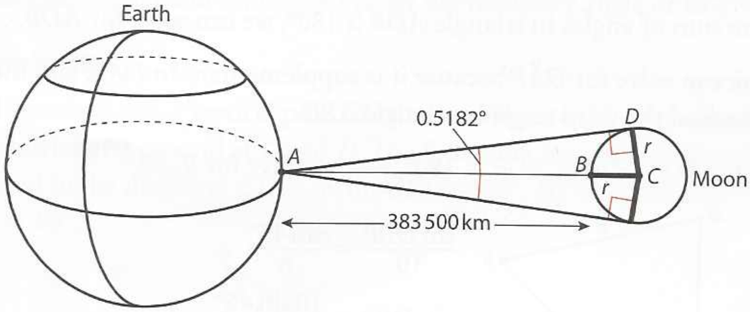
Then, $h \approx 524.795 - 470 \approx 54.795$ m.

Therefore, the height of the lighthouse is 54.8 m.

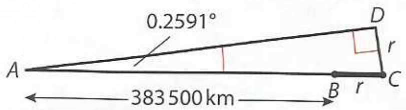
Example 28

As viewed from the surface of the Earth, the angle subtended by the full Moon is 0.5182°. Given that the distance from the Earth's surface to the Moon's surface is approximately 383 500 km, find the radius of the Moon to 3 s.f.

Solution



Remember that the radius of a circle drawn to a point of tangency will be perpendicular to the tangent line. This gives us two right triangles in the diagram – each with one acute angle having a measure of $\frac{1}{2}(0.5182°) = 0.2591°$. Extract right triangle ADC from the diagram.



$\sin(0.2591°) = \dfrac{r}{383\,500 + r}$

$r = (383\,500 + r)\sin(0.2591°)$

$r = 383\,500\sin(0.2591°) + r\sin(0.2591°)$

$r - r\sin(0.2591°) = 383\,500\sin(0.2591°)$ Collect terms containing r on the left side.

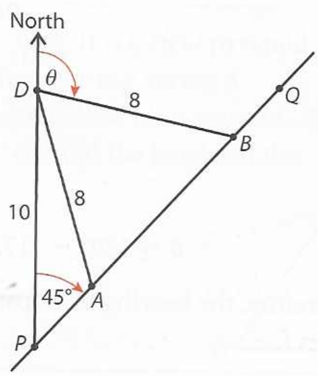
$r(1-\sin(0.2591°)) = 383\,500\sin(0.2591°)$ Factor out r from the expression on the left side.

$r = \dfrac{383\,500\sin(0.2591°)}{1 - \sin(0.2591°)} \approx 1742.12$ km

Therefore, the approximate radius of the Moon is 1740 km to 3 s.f.

Example 29

The diagram shows a point P that is 10 km due south of a point D. A straight road PQ is such that the (compass) bearing of Q from P is 45°. A and B are two points on this road which are both 8 km from D. Find the bearing of B from D, approximated to 3 s.f.



Solution

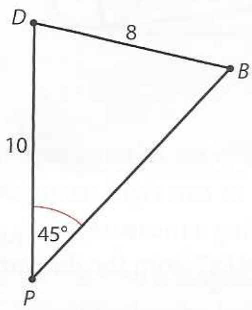
The angle θ in the diagram is the bearing ***of** B **from** D*. A strategy that will lead to finding θ is:

(1) Extract triangle PDB and use the law of sines to solve for $D\hat{B}P$.

(2) Triangle ADB is isosceles (two sides equal), so $D\hat{A}B = D\hat{B}P$; and since the sum of angles in triangle ADB is 180°, we can solve for $A\hat{D}B$.

(3) We can solve for $D\hat{A}P$ because it is supplementary to $D\hat{A}B$, and then we can find the third angle in triangle APD.

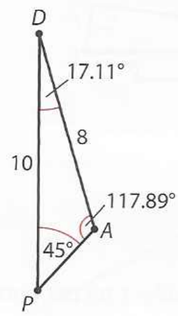
(4) Since $\theta + A\hat{D}B + A\hat{D}P = 180°$, we can solve for θ.

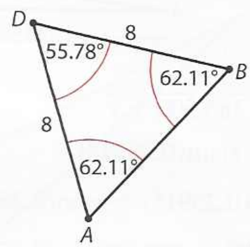


$$\frac{\sin D\hat{B}P}{10} = \frac{\sin 45°}{8}$$

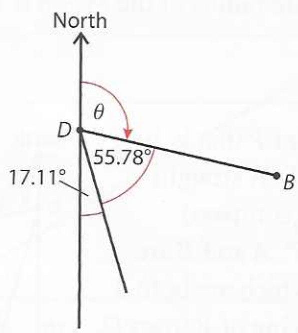
$$\sin D\hat{B}P = \frac{10\sin 45°}{8}$$

$$D\hat{B}P = \sin^{-1}\left(\frac{10\sin 45°}{8}\right) \approx 62.11°$$





$$D\hat{A}B = D\hat{B}P \approx 62.11°$$
$$A\hat{D}B \approx 180° - 2(62.11°) \approx 55.78°$$
$$P\hat{A}D \approx 180° - 62.11° \approx 117.89°$$
$$A\hat{D}P \approx 180° - (45° + 117.89°) \approx 17.11°$$



Compass bearings are measured **clockwise** from north.

$$\theta \approx 180° - (17.11° + 55.78°) \approx 107.11°$$

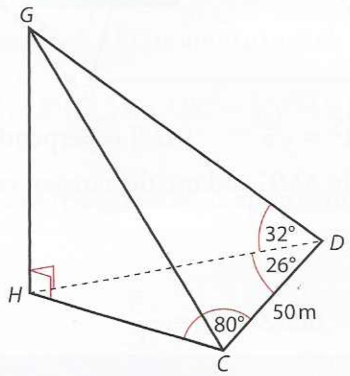
Therefore, the bearing of B from D is approximately 107° to an accuracy of 3 s.f.

Three-dimensional trigonometry problems

Of course, not all applications of triangle trigonometry are restricted to just two dimensions. In many problems, it is necessary to calculate lengths and angles in three-dimensional structures. As in the preceding section, it is very important to carefully analyze the three-dimensional diagram and to extract any relevant triangles in order to solve for the necessary angle or length.

Example 30

The diagram shows a vertical pole GH that is supported by two wires fixed to the horizontal ground at C and D. The following measurements are indicated in the diagram: $CD = 50$ m, $G\hat{D}H = 32°$, $H\hat{D}C = 26°$ and $H\hat{C}D = 80°$.



Find a) the distance between H and D, and b) the height of the pole GH.

Solution

a) In triangle HDC: $D\hat{H}C = 180° - (80° + 26°) = 74°$.

Now apply the law of sines:

$$\frac{\sin 80°}{HD} = \frac{\sin 74°}{50} \Rightarrow HD = \frac{50 \sin 80°}{\sin 74°} \approx 51.225\,\text{m}$$

Therefore, the distance from H to D is 51.2 m accurate to 3 s.f.

b) Using the right triangle GHD:

$$\tan 32° = \frac{GH}{51.225} \Rightarrow GH = 51.225 \tan 32° \approx 32.009\,\text{m}$$

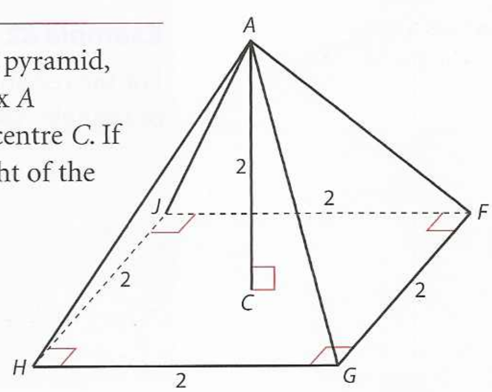
Therefore, the height of the pole is 32.0 m accurate to 3 s.f.

Example 31

The figure shown is a pyramid with a square base. It is a *right* pyramid, so the line segment (i.e. the height) drawn from the top vertex A perpendicular to the base will intersect the square base at its centre C. If each side of the square base has a length of 2 cm and the height of the pyramid is also 2 cm, find:

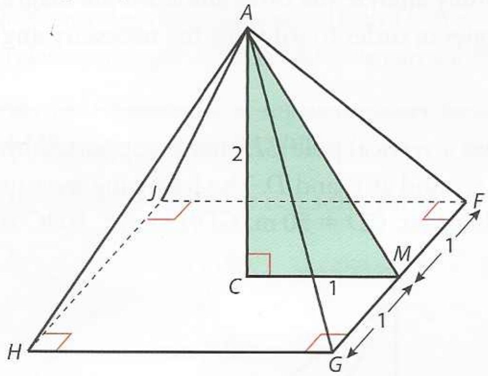
a) the measure of $A\hat{G}F$

b) the total surface area of the pyramid.



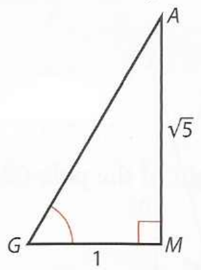
Solution

a) Label the midpoint of $[GF]$ as point M and draw two line segments, $[CM]$ and $[AM]$. Since C is the centre of the square base then $CM = 1$ cm. Extract right triangle ACM and use Pythagoras' theorem to find the length of $[AM]$.



$AM = \sqrt{1^2 + 2^2} = \sqrt{5}$ $[AM]$ is perpendicular to $[GF]$

Extract right triangle AMG and use the tangent ratio to find $A\hat{G}M$ (same as $A\hat{G}F$):



$$\tan(A\hat{G}M) = \frac{\sqrt{5}}{1}$$
$$A\hat{G}M = \tan^{-1}(\sqrt{5}) \approx 65.905°$$

Therefore, $A\hat{G}M = A\hat{G}F \approx 65.9°$.

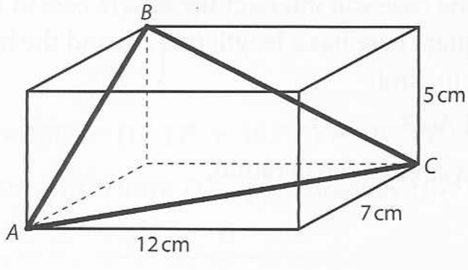
b) The total surface area comprises the square base plus four identical lateral faces that are all isosceles triangles. Triangle AGM is one-half the area of one of these triangular faces.

$$\text{Area of triangle } AGM = \frac{1}{2}(1)(\sqrt{5}) = \frac{\sqrt{5}}{2}$$
$$\Rightarrow \text{Area of triangle } AGF = 2\left(\frac{\sqrt{5}}{2}\right) = \sqrt{5}$$

$$\begin{aligned}\text{Surface area} &= \text{area of square base} + \text{area of four lateral faces}\\ &= 2^2 + 4\sqrt{5} = 4 + 4\sqrt{5} \approx 12.94\text{ cm}^2\end{aligned}$$

Example 32

For the rectangular box shown, find a) the measure of $A\hat{B}C$, and b) the area of triangle ABC.



Solution

a) Each of the three sides of triangle ABC is the hypotenuse of a right triangle. Using Pythagoras' theorem:

$$AC = \sqrt{7^2 + 12^2} = \sqrt{49 + 144} = \sqrt{193} = 13.892$$

$$AB = \sqrt{5^2 + 7^2} = \sqrt{25 + 49} = \sqrt{74} \approx 8.602$$

$$BC = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

Apply the law of cosines to find $A\hat{B}C$, using exact lengths of the sides of the triangle.

$$\cos A\hat{B}C = \frac{(\sqrt{74})^2 + 13^2 - (\sqrt{193})^2}{2(\sqrt{74})(13)} \Rightarrow A\hat{B}C = \cos^{-1}\left[\frac{74 + 169 - 193}{2(\sqrt{74})(13)}\right] \approx 77.082°$$

Therefore, the measure of $A\hat{B}C$ is approximately 77.1° to 3 s.f.

b) Area of triangle $= \frac{1}{2}(AB)(BC)\sin A\hat{B}C = \frac{1}{2}(\sqrt{74})(13)\sin(77.082°)$
$\approx 54.499\,96\text{ cm}^2$

Therefore, the area of triangle ABC is approximately 54.5 cm^2.

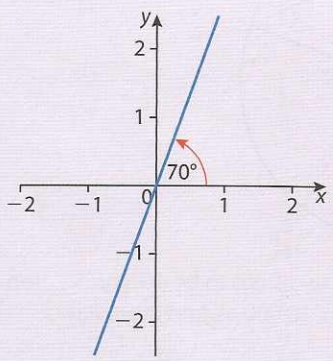
Exercise 7.5

In questions 1–4, determine:

a) the slope (gradient) of the line (approximate to 3 s.f. if not exact)

b) the equation of the line.

1



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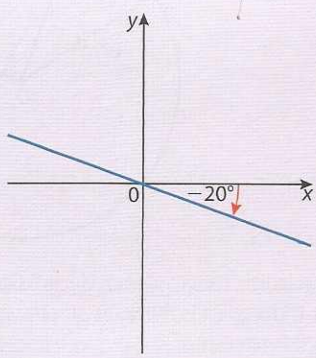

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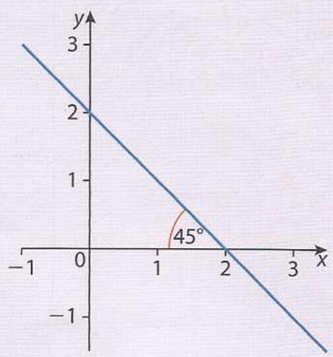

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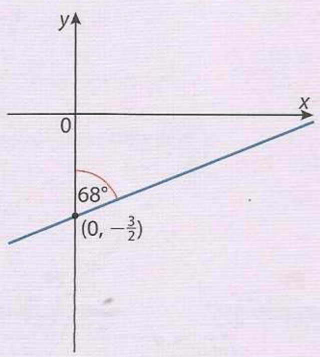


In questions 5–7, find the acute angle that the line through the given pair of points makes with the x-axis.

5 (1, 4) and (−1, 2)

6 (−3, 1) and (6, −5)

7 $\left(2, \frac{1}{2}\right)$ and (−4, −10)

In questions 8 and 9, find the acute angle between the two given lines.

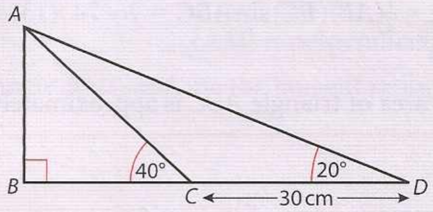
8 $y = -2x$ and $y = x$

9 $y = -3x + 5$ and $y = 2x$

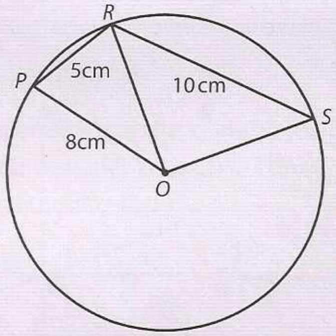
10 a) Find the exact equation of line L_1 that passes through the origin and makes an angle of 30° with the positive direction of the x-axis.

b) The equation of line L_2 is $x + 2y = 6$. Find the acute angle between L_1 and L_2.

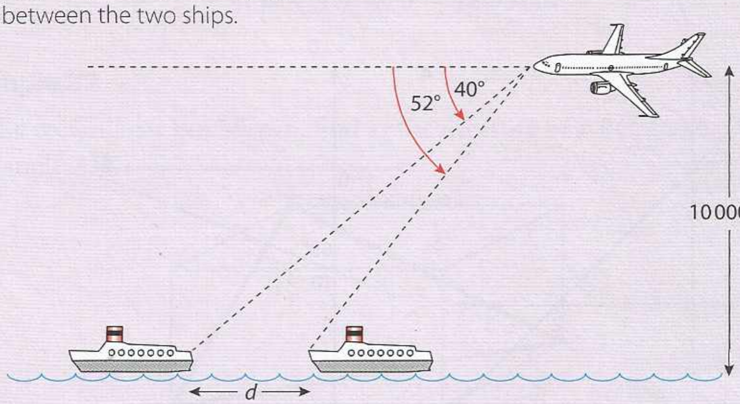
11 Calculate AB given $CD = 30$ cm, and the angle measures given in the diagram.



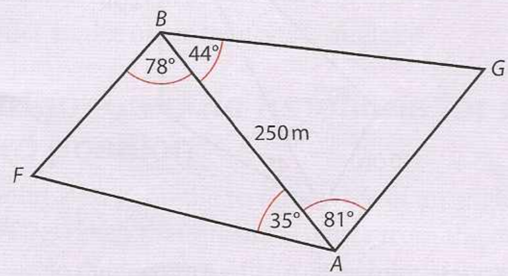
12 The circle with centre O and radius of 8 cm has two chords PR and RS, such that $PR = 5$ cm and $RS = 10$ cm. Find each of the angles $P\hat{R}O$ and $S\hat{R}O$, and then calculate the area of the triangle PRS.



13 A pilot measures the angles of depression to two ships to be 40° and 52° (see diagram). If the pilot is flying at an elevation of 10 000 m, find the distance between the two ships.



14 A forester was conducting a survey of a tropical jungle that was mostly inaccessible on foot. The points *F* and *G* indicate the location of two rare trees. To find the distance between points *F* and *G*, a line *AB* of length 250 m is measured out so that *F* and *G* are on opposite sides of *AB*. The angles between the line segment *AB* and the line of sight from each endpoint of *AB* to each tree are measured, and are shown in the diagram. Calculate the distance between *F* and *G*.

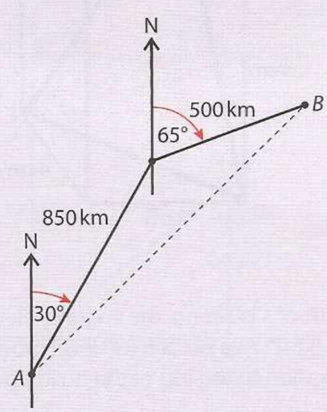


15 Calculate the distance between the tips of the hands of a large clock on a building at 10 o'clock if the minute hand is 3 m long and the hour hand is 2.25 m long.

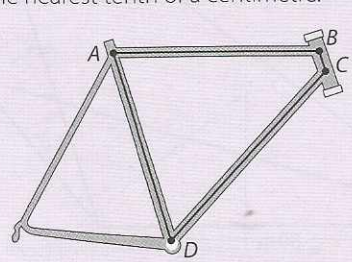
16 An airplane takes off from point *A*. It flies 850 km on a bearing of 030°. It then changes direction to a bearing of 065° and flies a further 500 km and lands at point *B*.

a) What is the straight line distance from A *to* B?

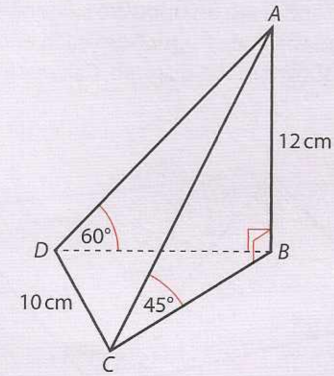
b) What is the bearing **from** *A* **to** *B*?



17 The traditional bicycle frame consists of tubes connected together in the shape of a triangle and a quadrilateral (four-sided polygon). In the diagram, *AB*, *BC*, *CD* and *AD* represent the four tubes of the quadrilateral section of the frame. A frame maker has prepared three tubes such that $AD = 53$ cm, $AB = 55$ cm and $BC = 11$ cm. If $D\hat{A}B = 76°$ and $A\hat{B}C = 97°$, what must be the length of tube *CD*? Give your answer to the nearest tenth of a centimetre.

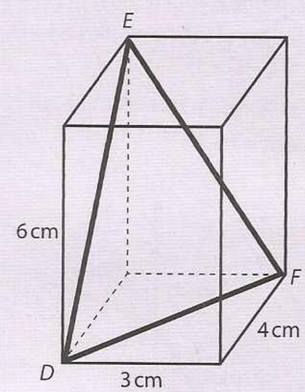


18 The tetrahedron shown in the diagram has the following measurements.
$AB = 12$ cm, $DC = 10$ cm, $A\hat{C}B = 45°$ and $A\hat{D}B = 60°$

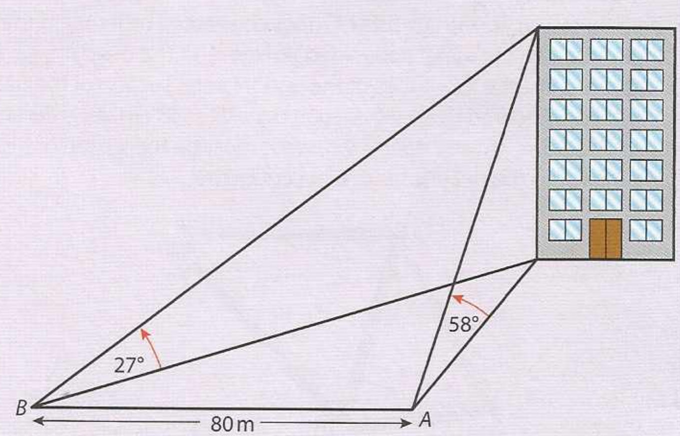


AB is perpendicular to the triangle BCD. Find the area of each of the four triangular faces: ABC, ABD, BCD and ACD.

19 Find the measure of angle DEF in the rectangular box.

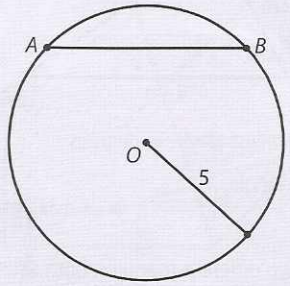


20 At a point A, due south of a building, the angle of elevation from the ground to the top of a building is 58°. At a point B (on level ground with A), 80 m due west of A, the angle of elevation to the top of the building is 27°. Find the height of the building.



Practice questions

1 The shortest distance from a chord $[AB]$ to the centre O of a circle is 3 units. The radius of the circle is 5 units. Find the exact value of $\sin A\hat{O}B$.



2 In a right triangle, $\tan\theta = \frac{3}{7}$. Find the exact value of $\sin 2\theta$ and $\cos 2\theta$.

3 A triangle has sides of length 4, 5 and 7 units. Find, to the nearest tenth of a degree, the size of the largest angle.

4 If A is an obtuse angle in a triangle and $\sin A = \frac{5}{13}$, calculate the exact value of $\sin 2A$.

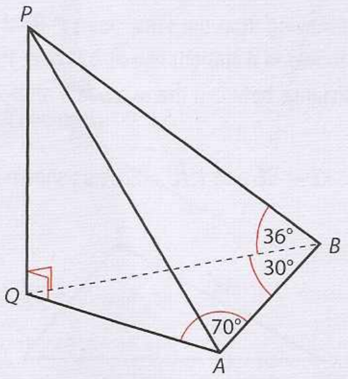
5 The diagram shows a vertical pole PQ, which is supported by two wires fixed to the horizontal ground at A and B.

$BQ = 40\,\text{m}$

$P\hat{B}Q = 36°$

$B\hat{A}Q = 70°$

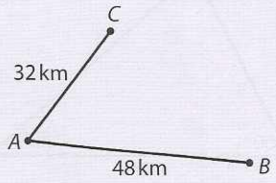
$A\hat{B}Q = 30°$



Find: **a)** the height of the pole PQ

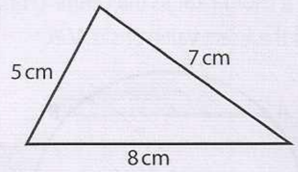
b) the distance between A and B.

6 Town A is 48 km from town B and 32 km from town C, as shown in the diagram.



Given that town B is 56 km from town C, find the size of the angle $C\hat{A}B$ to the nearest tenth of a degree.

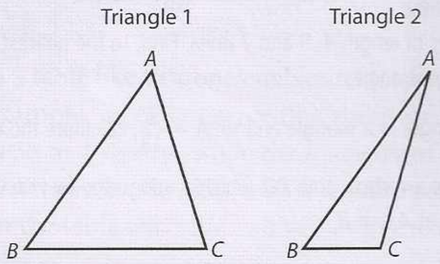
7 The following diagram shows a triangle with sides 5 cm, 7 cm and 8 cm.



Find: **a)** the size of the smallest angle, in degrees

b) the area of the triangle.

8 The diagrams below show two different triangles, both satisfying the conditions: $AB = 20$ cm, $AC = 17$ cm, $A\hat{B}C = 50°$.

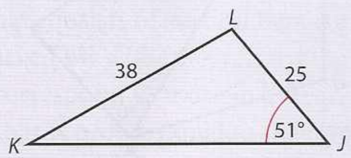


a) Calculate the size of $A\hat{C}B$ in Triangle 2.

b) Calculate the area of Triangle 1.

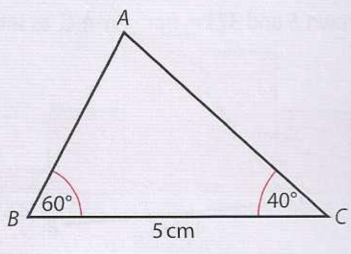
9 Two boats A and B start moving from the same point P. Boat A moves in a straight line at 20 km/h and boat B moves in a straight line at 32 km/h. The angle between their paths is 70°. Find the distance between the two boats after 2.5 hours.

10 In triangle JKL, $JL = 25$, $KL = 38$ and $K\hat{J}L = 51°$, as shown in the diagram.



Find $J\hat{K}L$, giving your answer correct to the nearest degree.

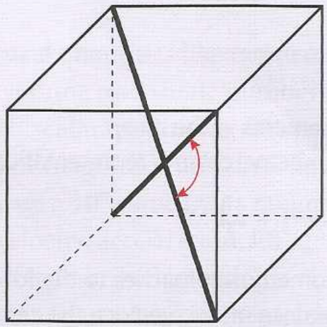
11 The following diagram shows a triangle ABC, where $BC = 5$ cm, $A\hat{B}C = 60°$ and $A\hat{C}B = 40°$.



a) Calculate AB.

b) Find the area of the triangle.

12 Find the measure of the acute angle between a pair of diagonals of a cube.

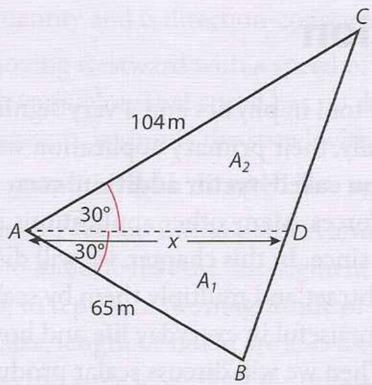


13 A farmer owns a triangular field *ABC*. One side of the triangle, [*AC*], is 104 m, a second side, [*AB*], is 65 m and the angle between these two sides is 60°.

a) Use the cosine rule to calculate the length of the third side, [*BC*], of the field.

b) Given that $\sin 60° = \frac{\sqrt{3}}{2}$, find the area of the field in the form $p\sqrt{3}$, where *p* is an integer.

Let *D* be a point on [*BC*] such that [*AD*] bisects the 60° angle. The farmer divides the field into two parts, A_1 and A_2, by constructing a straight fence [*AD*] of length *x* m, as shown in the diagram.



c) **(i)** Show that the area of A_1 is given by $\frac{65x}{4}$.

(ii) Find a similar expression for the area of A_2.

(iii) Hence, find the value of *x* in the form $q\sqrt{3}$, where *q* is an integer.

d) **(i)** Explain why $\sin A\hat{D}C = \sin A\hat{D}B$.

(ii) Use the result of part **(i)** and the sine rule to show that $\frac{BD}{DC} = \frac{5}{8}$.