L2 - Completing the Square

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Quadratic Functions
Lesson 2 Completing the Square

provided

When a quadratic function is given in **vertex form** $y = a(x-p)^2 + q$, we know:

- (p,q) = vertex *p is the opposite of how it looks
- x = p axis of symmetry
- a = stretch → opens up/down

Today we will learn how to graph a quadratic function given in **general form** $y = ax^2 + bx + c$ (y-int; stretch)

Eg1. Expand the following.

a) $(x+3)^2$ = (x + 3) (x + 3)

= x · x + 3 · x + 3 · x + 3 · 3

= $x^2 + 2(3x) + 3^2$

= $x^2 + 6x + 9$

b) $(k-7)^2$ = (k − 7)(k − 7)

= $k^2 - 7k - 7k + 49$

= $k^2 - 14k + 49$

How does the middle term of the expanded form relate to the second term of the binomial?

They are twice as big.

To transform a quadratic into a perfect square, we take a half of the "x" term.

Eg2. Complete the squares for the following. State the vertex. → Rewrite in vertex form!

a) $y = x^2 + 6x - 5$

$\left[\frac{1}{2}(6)\right]^2 = 9$

$y = x^2 + 6x + 9 - 9 - 5$ (factor)

$y = (x+3)^2 - 14$

→ V: (−3, −14)

b) $y = x^2 - 2x + 8$

$\left[\frac{1}{2}(-2)\right]^2 = 1$

$y = x^2 - 2x + 1 - 1 + 8$

$y = (x-1)^2 + 7$

→ V: (1, 7)

If there is a leading coefficient (a) of the quadratic function, we need to factor it out.

Eg3. Complete the squares for the following. Then graph the quadratic function.

a) $y = -2x^2 + 4x - 3$

$\left[\frac{1}{2}(-2)\right]^2$ $y = a(x-p)^2 + q$

*factor 'a' from x^2, x only

$y = -2(x^2 - 2x) - 3$

$y = -2(x^2 - 2x + 1 - 1) - 3$

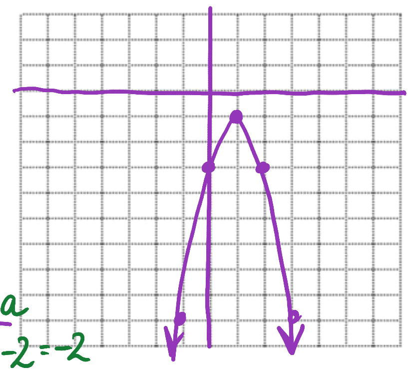
$y = -2(x^2 - 2x + 1) + 2 - 3$

$y = -2(x-1)^2 - 1$

vertex: $(1, -1)$

stretch: -2

	$x^2 \times a$
1	$1 \times -2 = -2$
2	$4 \times -2 = -8$



b) $y = 4x^2 - 9x + 1$

$y = 4(x^2 - \frac{9}{4}x) + 1$ $\left[\frac{1}{2}\left(-\frac{9}{4}\right)\right]^2 = \left[-\frac{9}{8}\right]^2 = \frac{81}{64}$

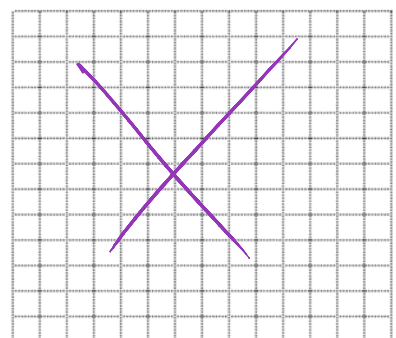
$= 4(x^2 - \frac{9}{4}x + \frac{81}{64} - \frac{81}{64}) + 1$ $4\left(\frac{-81}{64}\right) = \frac{-81}{16}$

$= 4(x^2 - \frac{9}{4}x + \frac{81}{64}) - \frac{81}{16} + \frac{16}{16}$

half

$y = 4(x - \frac{9}{8})^2 - \frac{65}{16}$ → vertex: $\left(\frac{9}{8}, \frac{-65}{16}\right)$

stretch: 4



Practices: Worksheet 3 # (1, 3, 4)bf, 5(i, iii, iv), 6, (7, 8)af, 10

EXERCISES

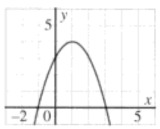
A 1. Write each equation in the form $f(x) = a(x - p)^2 + q$. Check by expanding the result.

a) $f(x) = x^2 - 6x + 8$ **b)** $f(x) = x^2 + 10x + 14$
c) $f(x) = 2x^2 + 4x + 7$ **d)** $f(x) = -2x^2 + 4x + 5$
e) $f(x) = 3x^2 - 24x + 40$ **f)** $f(x) = -5x^2 - 20x - 30$

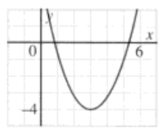
2. Sketch the graph of each parabola in exercise 1.

3. Use the word "maximum" or "minimum," and data from each graph. Copy and complete this sentence for each graph:
"The … value of y is … when $x =$ …"

a)



b)



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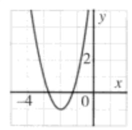

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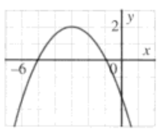
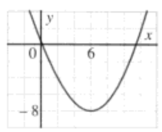


e)



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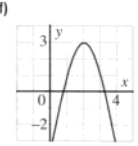

4. Does each function have a maximum value? If it has, for what value of x does it occur?

a) $y = -2(x + 5)^2 - 8$ **b)** $f(x) = \frac{1}{4}(x - 2)^2 - 9$ **c)** $y = -0.5(x - 3)^2 + 7.5$
d) $y = 5 - 3x^2$ **e)** $f(x) = 3\left(x - \frac{5}{2}\right)^2 + \frac{17}{2}$ **f)** $f(x) = -(x + 4)^2 - 19$

5. **a)** Write each equation in the form $y = a(x - p)^2 + q$.

i) $y = 2x^2 - 8x + 15$ **ii)** $y = 3x^2 + 12x - 7$ **iii)** $y = x^2 - 6x + 7$
iv) $y = -2x^2 + 6x + 11$ **v)** $y = -x^2 - 3x - 3$ **vi)** $y = 1.5x^2 - 9x + 10$

b) For each function in part a:
i) State its maximum or minimum value.
ii) State the value of x for which the maximum or minimum occurs.

6. How is the constant a in the equation $y = ax^2 + bx + c$ related to the constant a in the equation $y = a(x - p)^2 + q$? Explain.

B 7. Sketch each parabola.
i) Label the vertex with its coordinates.
ii) Label the axis of symmetry with its equation.
iii) Label two points on the graph with their coordinates.

a) $f(x) = x^2 - 6x + 10$ **b)** $y = 2x^2 + 8x + 7$ **c)** $y = -x^2 + 10x - 13$
d) $f(t) = 3t^2 - 6t + 8$ **e)** $f(n) = -4n^2 - 24n - 20$ **f)** $u = -2v^2 - 16v - 35$

8. Sketch each parabola.
i) Label the vertex with its coordinates.
ii) Label the axis of symmetry with its equation.
iii) Label two other points on the graph with their coordinates.

a) $y = \frac{1}{2}x^2 - 2x + 7$ **b)** $f(t) = 4t^2 + 12t - 5$ **c)** $f(j) = -2j^2 + 14j - 12$
d) $y = 3x^2 - 4x - 6$ **e)** $u = -4v^2 + 10v - 7$ **f)** $f(x) = -2x^2 - 12x - 14$

9. Choose one parabola from exercise 7 or 8. Write to explain how you sketched it.

10. For each parabola, state:
i) the maximum or minimum value of y
ii) whether it is a maximum or minimum
iii) the value of x when it occurs
iv) the domain and range of the function

a) $y = (x - 3)^2 + 5$ **b)** $y = 2(x + 1)^2 - 3$ **c)** $y = -2(x - 1)^2 + 4$
d) $y = -(x + 2)^2 - 6$ **e)** $y = 0.5x^2 - 9$ **f)** $y = 7 - 2x^2$

11. A company manufactures and sells designer T-shirts. The profit, P dollars, for selling a certain style of T-shirt is projected to be $P = -20x^2 + 1000x - 6720$, where x dollars is the selling price of one T-shirt.

a) What selling price gives the maximum profit? What is the maximum profit?
b) The company hopes to earn a profit in excess of \$6000 on this style of T-shirt. Based on its projections, is this possible?
c) Sketch a graph of this function.

MODELLING T-Shirt Price Increases

In exercise **11.** the equation $P = -20x^2 + 1000x - 6720$ models the projected profit, P dollars, as a quadratic function of the selling price, x dollars.

- Suggest how the company might have determined this equation, or one similar.
- What is a reasonable domain for the function?
- What do the x-intercepts of the graph represent? What does the y-intercept represent?
- Why is a parabolic graph that opens down a reasonable model for this situation?

12. On a forward somersault dive, a diver's height, h metres, above the water is given by $h(t) = -4.9t^2 + 6t + 3$, where t is the time in seconds after the diver leaves the board.

a) Graph the function.
b) Determine the diver's maximum height above the water.
c) How long does it take the diver to reach the maximum height?
d) For how long is the diver higher than 3 m above the water?

Exercises

1. a) $f(x) = (x-3)^2 - 1$ **b)** $f(x) = (x+5)^2 - 11$
c) $f(x) = 2(x+1)^2 + 5$ **d)** $f(x) = -2(x-1)^2 + 7$
e) $f(x) = 3(x-4)^2 - 8$ **f)** $f(x) = -5(x+2)^2 - 10$

3. a) The maximum value of y is 4 when x is 1.
b) The minimum value of y is –4 when x is 3.
c) The minimum value of y is –1 when x is –2.
d) The maximum value of y is 2 when x is –3.
e) The minimum value of y is –8 when x is 6.
f) The maximum value of y is 3 when x is 2.

4. a) Yes; $y = -8$ when $x = -5$ **b)** No
c) Yes; $y = 7.5$ when $x = 3$
d) Yes; $y = 5$ when $x = 0$ **e)** No
f) Yes; $y = -19$ when $x = -4$

5. a) i) $y = 2(x-2)^2 + 7$ **ii)** $y = 3(x+2)^2 - 19$
iii) $y = (x-3)^2 - 2$ **iv)** $y = -2(x-1.5)^2 + 15.5$
v) $y = -1(x+1.5)^2 - 0.75$ **vi)** $y = 1.5(x-3)^2 - 3.5$
b) i) i) 7, minimum **ii)** 2
ii) i) –19, minimum **ii)** –2
iii) i) –2, minimum **ii)** 3
iv) i) 15.5, maximum **ii)** 1.5
v) i) –0.75, maximum **ii)** –1.5
vi) i) –3.5, minimum **ii)** 3

6. The values of a are equal. Explanations may vary.

9. Explanations may vary.

10. a) i) 5 **ii)** Minimum **iii)** 3
iv) D: all real numbers; R: $y \geq 5$
b) i) –3 **ii)** Minimum **iii)** –1
iv) D: all real numbers; R: $y \geq -3$
c) i) 4 **ii)** Maximum **iii)** 1
iv) D: all real numbers; R: $y \leq 4$
d) i) –6 **ii)** Maximum **iii)** –2
iv) D: all real numbers; R: $y \leq -6$
e) i) –9 **ii)** Minimum **iii)** 0
iv) D: all real numbers; R: $y \geq -9$
f) i) 7 **ii)** Maximum **iii)** 0
iv) D: all real numbers; R: $y \leq 7$

11. a) \$25; \$5780 **b)** No, maximum profit is \$5780.

12. b) 4.84 m **c)** 0.61 s **d)** About 1.2 s