Notes

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9:46 AM

1.1: Arithmetic Sequences

What do you think of when you hear the word **sequence**?

repeated pattern, order, eg. 1, 2, 3, 4, ..
up or down

A **sequence** is: a list with a pattern.

- A **finite sequence** has a set number of terms. Ex. 1, 2, 3
- An **infinite sequence** has an infinite number of terms. Ex. 1, 2, 3, ...

Arithmetic sequences have a common difference (d) between any two terms.

eg1. 3, 5, 7, 9, 11 — t_1, +2, $d=2$

eg2. $-10, -15, -20, -25, \ldots$ — t_1, −5, $d=-5$

The **general term** of an arithmetic sequence is: $\boxed{t_n = t_1 + (n-1)d}$

Where:
- t_n = n^{th} term (usually stopping point)
- t_1 = 1^{st} term
- n = position in the sequence
- d = common difference $[d = t_2 - t_1]$

Ex.1: Are the following arithmetic sequences? If so, write the general term t_n.

a) 34, 41, 48, 55,... (+7 +7 +7)

Yes!

$t_n = t_1 + (n-1)d$

$t_n = 34 + (n-1)7$

$\boxed{t_n = 7n + 27}$

b) -0.5, -14, - 27.5,... (−13.5 −13.5)

$-14-(-0.5)$ Yes!

$t_n = -0.5 + (n-1)(-13.5)$

$\boxed{t_n = -13.5n + 13}$

Ex.2: Determine the missing terms of the following sequence if it is arithmetic:

must have 'd'.

6, 4.5, 3, $\frac{3}{2}$

$d = \frac{3}{2} - \frac{\cancel{3}6}{2} = -\frac{3}{2}$ or -1.5

1

Ex.3: In an arithmetic sequence, $t_{21} = -1.4$ and $d = 0.6$. Find the first three terms and the general term t_n

n t_n

$t_n = t_1 + (n-1)d$

$d = 0.6$ $-1.4 = t_1 + (21-1)0.6$
$n = 21$ $-1.4 = t_1 + (20)0.6$
$t_n = -1.4$ $-1.4 = t_1 + 12$

$t_1 = -13.4$

1st 3 terms: $-13.4, -12.8, -12.2$ (+0.6, +0.6)

General Term: $t_n = -13.4 + (n-1)0.6$
$t_n = 0.6n - 14$

Ex.4: In an arithmetic sequence, $t_{12} = 52$ and $t_{22} = 102$. Find the first three terms of the sequence and the general term t_n.

System!

$t_n = t_1 + (n-1)d$

① $52 = t_1 + (12-1)d$ ② $102 = t_1 + (22-1)d$
$52 = t_1 + 11d$ $102 = t_1 + 21d$

Subst. or Elim.

$-(52 = t_1 + 11d)$

$50 = 10d$

$d = 5$

① $52 = t_1 + 11(5)$
$-3 = t_1$

1st 3 terms: $-3, 2, 7$ (+5, +5)

General term: $t_n = -3 + (n-1)5$
$t_n = 5n - 8$

Ex.5: The numbers x, y, and z are the first three terms of an arithmetic sequence. Express z in terms of x and y.

x, y, z
+d +d

$d = y - x$ or $d = z - y$

=

$y - x = z - y$

$\boxed{2y - x = z}$

1.2: Arithmetic Series

(+) $2+8+14+20+26$

A **series** is the sum of the terms of a sequence. If t_n is a sequence, then the sum of the first n terms of the sequence is the series S_n.

← Sum

- The first Mathematician to discover a way to find the sum of a series was Carl Friedrich Gauss when he was in elementary school!
- **Gauss Problem**: Sum the integers from 1 to 100

$$\begin{array}{l} 1+2+\dots+50 \\ 100+99+\dots+51 \\ \hline \underbrace{101+101+\dots+101}_{50 \text{ pairs}} \end{array}$$

$50 \times 101 = 5050$

Half # of terms

First term + Last term

Sum of an Arithmetic Series:

$$\boxed{S_n = \frac{n}{2}(t_1 + t_n)}$$

this formula can be re-written using the general form from yesterday: use if given t_n

$t_n = t_1 + (n-1)d$

$S_n = \frac{n}{2}[t_1 + t_1 + (n-1)d]$

$S_n = \frac{n}{2}[2t_1 + (n-1)d]$

use if given d

$$\boxed{S_n = \frac{n}{2}[2t_1 + (n-1)d]}$$

Ex.1: Determine the sum of the first 20 terms of the arithmetic series 204 + 212 + 220 + …

$d = 8$

$t_1 = 204$

$n = 20$

$S_{20} = \frac{20}{2}[2(204) + (20-1)8]$

$\boxed{S_{20} = 5600}$

Ex.2: Determine the sum of the following arithmetic series: 7+14+21+…+98

$d = 7$

$t_1 = 7$

$t_n = 98$

$n = ?$

$S_n = ?$

Find n: $t_n = t_1 + (n-1)d$

$98 = 7 + (n-1)7$

$91 = (n-1)7$

$13 = n-1$

$n = 14$

$S_{14} = \frac{14}{2}(7+98)$

$\boxed{S_{14} = 735}$

Ex.3: Determine the number of terms n in the arithmetic series given $t_1 = 19$, $t_n = 96$, $S_n = 690$

$\underbrace{19 + \dots + 96}_{n=?} = 690$

$S_n = \frac{n}{2}(t_1 + t_n)$

$2 \times 690 = \frac{n}{\cancel{2}}(19+96) \times \cancel{2}$

$1380 = 115n$

n is always a whole number!

$\boxed{n=12}$

Ex.4: The sum of the first 2 terms in an arithmetic series is 13 and the sum of the first 4 terms is 46. Determine the first 3 terms of the series and their sum.

$t_1 + t_2 = 13 \rightarrow S_2 = 13$

$t_1+t_2+t_3+t_4 = 46 \rightarrow S_4 = 46$

} use $S_n = \frac{n}{2}[2t_1 + (n-1)d]$ to create a system.

① $13 = \frac{\cancel{2}}{\cancel{2}}[2t_1 + (2-1)d]$ ② $46 = \frac{4}{2}[2t_1 + (4-1)d]$

$13 = 2t_1 + d$

$\frac{46}{2} = \frac{\cancel{2}}{\cancel{2}}[2t_1 + 3d]$

$23 = 2t_1 + 3d$

$-\ 13 = 2t_1 + d$

$10 = 2d \rightarrow d = 5$

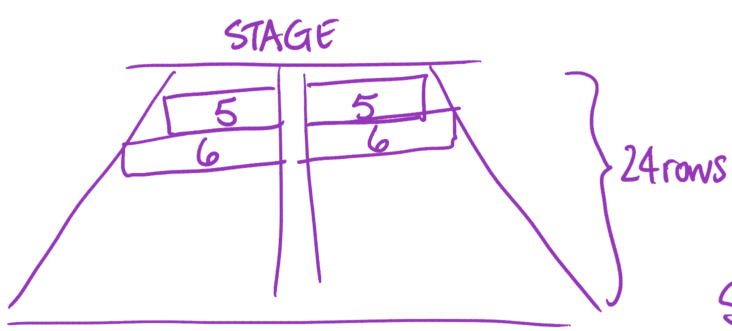
Elim.

$13 = 2t_1 + 5$

$t_1 = 4$

$\boxed{\begin{array}{c} 1^{st}\ 3\text{ terms:} \\ 4, 9, 14 \\ S_3 = 27 \end{array}}$

Ex.5: A theatre has 24 rows of seats. The front row has 5 seats on each side of the centre aisle. Each successive row has one more seat on each side. How many seats are altogether?



$10 + 12 + 14 + \dots$

$t_1 = 10$

$d = 2$

$n = 24$

$S_{24} = \frac{24}{2}[2(10) + (24-1)2]$

$\boxed{S_{24} = 792}$

1.3: Geometric Sequences

~~*Explore*~~: With a partner, repeatedly fold a sheet of paper in half. At each stage, record the number of layers of paper in our folded sheet.

Number of folds	Number of layers of paper
0	1
1	2
2	
3	
4	
5	

Consider the number of layers as a sequence.

➔ What is the next number in the sequence?

➔ How could you find the next number in general?

➔ What is the number of layers of paper after 10 folds?

➔ What expression could represent the *next* term in the sequence?

*multiplying

Geometric sequences have a common ratio (r) between any two terms.

eg1. 1, 2, 4, 8, 16 $r = 2$ ×2

eg2. −10, 5, −2.5, 1.25,.. $r = -\frac{1}{2}$ $\times -\frac{1}{2}$

How do we find it? $r = \frac{t_2}{t_1}$

The **general term** of a geometric sequence is: $\boxed{t_n = t_1 (r)^{n-1}}$

5

$\times 2$

Ex.1: Use the geometric sequence 5, 10, 20, 40, 80, …

a) Write the general term t_n of the sequence.

$t_n = t_1(r)^{n-1}$

$r = \frac{t_2}{t_1} = \frac{10}{5}$

$\boxed{t_n = 5(2)^{n-1}}$ can't be simplified!

b) Use the general term from part a) to determine the value of t_8.

$t_8 = 5(2)^{8-1}$

$t_8 = 5(2)^7$

$\boxed{t_8 = 640}$

Ex.2: After each washing, 1% of the dye in blue jeans is washed out. How much of the original dye remains after 10 washings?

% dye: 100, 99, 98.01, 97.0299, …

$t_1 = 100$

$r = \frac{99}{100} = 0.99$

$n = 11$ (after 10 washes)

$t_{11} = 100(0.99)^{11-1}$

$t_{11} = 90.44$

$\boxed{\text{There is 90.44\% dye left}}$

Ex.3: In 1990 the population of Canada was 26.6 million. The population in 2025 is projected to 38.4 million. If this projection were based on a geometric sequence, what would be the annual growth rate?

26.6, —, —, …, 38.4

$t_1 = 26.6$

$t_n = 38.4$

$n = 2025 - 1990 + 1$

$= 36$

include endpoint (1990)

$38.4 = 26.6(r)^{36-1}$

$\frac{38.4}{26.6} = \frac{\cancel{26.6}}{\cancel{26.6}}(r)^{35}$

$1.4436 = r^{35}$

$r = \sqrt[35]{1.4436}$

$\boxed{r = 1.01}$

$\boxed{\text{MATH}}$ $\boxed{5: \sqrt[x]{\ }}$

Ex.4: Two years after purchase, the resale value of a car was \$10 000. The resale value of the same car three years later was \$5000. If the annual depreciation of the car forms a geometric sequence, what was the original price of the car?

$\div r$ $\div r$ $\times r$

—, —, 10 000, —, —, 5000

$t_3 = 10\,000$

$t_6 = 5000$

① $10000 = t_1(r)^{3-1}$

$10000 = t_1 r^2$

② $5000 = t_1(r)^{6-1}$

$5000 = \cancel{t_1} r^5$

① $10000 = \cancel{t_1} r^2$ $\div$

$0.5 = r^3$

$r = 0.7937$

$t_1 = \frac{10000}{0.7937^2} = \boxed{15874.01}$

eg. $1 + 2 + 4 + 8 + 16 + \ldots$

1.4: Geometric Series

A **geometric series** is the sum of the terms of a geometric sequence.

The formula for the sum of a geometric series is: $\boxed{S_n = \dfrac{t_1(r^n - 1)}{r - 1}}$

Ex.1: Find the sum of the first 10 terms, S_{10}, of the geometric series $4+12+36+\ldots$

$t_1 = 4$

$r = 3$

$n = 10$

$$S_{10} = \frac{4(3^{10} - 1)}{3 - 1}$$

$$\boxed{S_{10} = 118\,096}$$

Ex.2: Find the sum of the following geometric series:

a) $\frac{1}{27} + \frac{1}{9} + \frac{1}{3} \ldots + 729$

$t_1 = \frac{1}{27}$

$r = \frac{1/9}{1/27} = \frac{1}{9} \times \frac{27}{1} = 3$

Use: $t_n = t_1 (r)^{n-1}$ to find n

$27 \times 729 = \frac{1}{\cancel{27}}(3)^{n-1} \times \cancel{27}$

Guess $3^?$ → $19683 = (3)^{n-1}$

$3^9 = (3)^{n-1} \rightarrow n = 10$

$$S_{10} = \frac{\frac{1}{27}(3^{10} - 1)}{3 - 1}$$

$$\boxed{= 1093.48}$$

b) $4 - 16 + 64 - \ldots - 65536$

$t_1 = 4$

$r = -4$

$t_n = t_1 (r)^{n-1}$

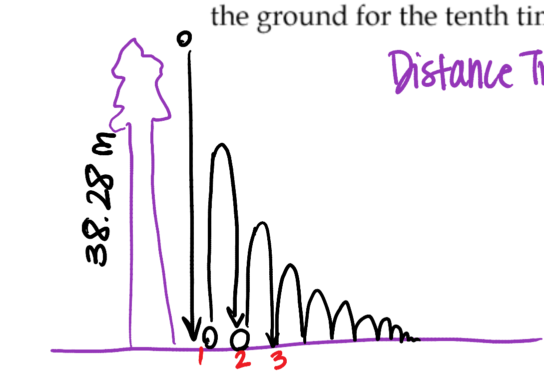
$\frac{-65536}{4} = \frac{\cancel{4}(-4)^{n-1}}{\cancel{4}}$

$-16384 = (-4)^{n-1}$

$(-4)^7 = (-4)^{n-1} \rightarrow n = 8$

$$S_8 = \frac{4((-4)^8 - 1)}{-4 - 1} = \boxed{-52428}$$

Ex.3: The tallest totem pole carved from a single log is 38.28 m high and is in Beacon Hill Park in Victoria, BC. If a lacrosse ball is dropped from this height and bounces back up 60% of the original height, find the total distance travelled by the ball by the time it hits the ground for the tenth time.



Distance Travelled: 38.28, 22.968, 13.7808, ...

down only; up + down; up + down

$t_1 = 38.28$

$r = 0.6$

$n = 10$

up + down; t_1 only down

$$S_{10} = 2\left[\frac{38.28(0.6^{10} - 1)}{0.6 - 1}\right] - 38.28$$

$$\boxed{S_{10} = 151.96 \text{ m}}$$

1.5: Infinite Geometric Series

Imagine that you are standing 100 m from a doorway. You approach the doorway by moving half the distance to the door in each motion for as long as possible. Write the sequence of distances moved in each step.

- What type of sequence is the pattern of steps? Why?

- How far have you moved toward the doorway after 3 steps? After 5 steps? Will you ever reach the doorway? Explain.

- What expression could you use to calculate the total distance that you have moved after 100 of these steps? Predict what approximate answer you would expect. If you were able to move infinite steps, what total distance would you have moved?

Ex.1: For the following geometric series $50+25+12.5+\dots$

a) Write a formula to express the sum for the series.

b) Using a calculator, find the sum if there are infinitely many terms.

Observation:

As __ becomes infinitely large, the value of t_n gets closer and closer to __________,

With $S_n = \dfrac{t_1\left(r^n - 1\right)}{r-1}$, what happens if $|r| < 1$ and n is large?

Ex.2: Find the sum of each infinite geometric series, if it exists.

a) $7+\frac{7}{2}+\frac{7}{4}+...$

b) $2-2+2-2+...$

Ex.3: Last month a well produced 15 000 m^3 of oil. Its production is known to be dropping by 2.9% each month.

a) How much oil will be produced over the next year, to three significant digits?

b) If the well is worked until it is dry, estimate what its total future production will be.

c) Actually, once the monthly production drops below 5000 m^3, it is not profitable to work the well. When should this well be capped?