Notes 2

January-22-15
9:30 AM

6.1: Rational Expressions

Objectives:
- determine non-permissible values for a rational expression
- simplify a rational expression

Rational Expressions: algebraic fractions with a numerator and a denominator that are both polynomials → contain variables.

- Can you come up with some examples?

$$\frac{x+5}{x^2-2} \quad , \quad \frac{7+y}{x^2+3} \quad , \quad \frac{x+y^2}{x^2-3} \quad , \quad \frac{5}{x}$$

What happens in each of the following expressions when $x = 3$ is substituted?

i) $\dfrac{x-7}{x-3} = \dfrac{-4}{0}$ ii) $\dfrac{x-7}{x^2-9} = \dfrac{-4}{0}$ iii) $\dfrac{x-7}{x^2-4x+3} = \dfrac{-4}{0}$

undefined when $x = 3$

What value(s) cannot be used for x in each of the following algebraic fractions?

a) $\dfrac{6-x}{2x}$ b) $\dfrac{3}{x-7}$ c) $\dfrac{4x-1}{(x-3)(2x+1)}$ → $2x+1 \neq 0$
                                                                        $x \neq \dfrac{-1}{2}$

$x \neq 0$ $x \neq 7$ $x \neq 3, \dfrac{-1}{2}$

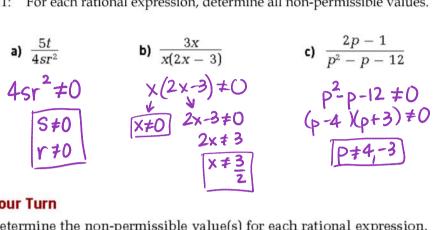What is the result when zero is divided by any non-zero number?

Why is *division by zero* **undefined**?

Try this! $\dfrac{1}{0.1} = 10$ $\dfrac{1}{0.01} = 100$ $\dfrac{1}{0.001} = 1000$ $\dfrac{1}{0.0001} = 10\,000$

$$\frac{1}{0.00000001} = 100\,000\,000$$

Denominator → getting smaller (close to 0).
Result → getting bigger
            approaching infinity!
                 undefined!

☐   ☐   ☐

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Whenever you use a rational expression, you must identify any values that must be excluded or are considered **non-permissible values**.

Non-permissible values: all values that make the denominator zero. (npv's)

Ex.1:   For each rational expression, determine all non-permissible values.

a) $\dfrac{5t}{4sr^2}$

$4sr^2 \neq 0$

$\boxed{\begin{array}{l} s \neq 0 \\ r \neq 0 \end{array}}$

b) $\dfrac{3x}{x(2x-3)}$

$x(2x-3) \neq 0$

$\boxed{x \neq 0}$ $2x - 3 \neq 0$

$2x \neq 3$

$\boxed{x \neq \dfrac{3}{2}}$

c) $\dfrac{2p-1}{p^2 - p - 12}$

$p^2 - p - 12 \neq 0$

$(p-4)(p+3) \neq 0$

$\boxed{p \neq 4, -3}$

Your Turn

Determine the non-permissible value(s) for each rational expression.

a) $\dfrac{4a}{3bc}$

$\boxed{\begin{array}{l} b \neq 0 \\ c \neq 0 \end{array}}$

b) $\dfrac{x-1}{(x+2)(x-3)}$

$\boxed{\begin{array}{l} x \neq -2 \\ x \neq 3 \end{array}}$

c) $\dfrac{2y^2}{y^2 - 4}$

$y^2 - 4 \neq 0$

$(y+2)(y-2) \neq 0$

$\boxed{y \neq -2, 2}$

Simplifying Rational Expressions

$$\boxed{\dfrac{9}{12} = \dfrac{\overset{1}{(3)}(3)}{(3)(4)} \underset{1}{} = \dfrac{3}{4}}$$

$$\boxed{\dfrac{m^3 t}{m^2 t^4} = \dfrac{\overset{1}{(m^2)}(m)\overset{1}{(t)}}{(m^2)(t)(t^3)\underset{1 \quad 1}{}} = \dfrac{m}{t^3}, \; m \neq 0, t \neq 0}$$
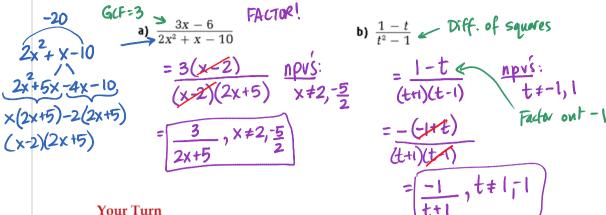
To simplify a rational expression:
- note any **non-permissible values** (npvs)
- divide both the numerator and denominator by any factors that are common to the numerator and the denominator.

* Factor top & bottom first!

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Ex.2:   Simplify each rational expression. State the non-permissible values.

GCF=3 → FACTOR!

a) $\dfrac{3x-6}{2x^2+x-10}$

-20

$2x^2 + x - 10$

$2x^2 + 5x - 4x - 10$

$x(2x+5) - 2(2x+5)$

$(x-2)(2x+5)$

$= \dfrac{3(x-2)}{(x-2)(2x+5)}$ npv's: $x \neq 2, -\dfrac{5}{2}$

$= \boxed{\dfrac{3}{2x+5}} \;, x \neq 2, -\dfrac{5}{2}$

b) $\dfrac{1-t}{t^2-1}$ ← Diff. of squares

$= \dfrac{1-t}{(t+1)(t-1)}$ npv's: $t \neq -1, 1$

Factor out -1

$= \dfrac{-(-1+t)}{(t+1)(t-1)}$

$= \boxed{\dfrac{-1}{t+1}} \;, t \neq 1, -1$

Your Turn

Simplify each rational expression. What are the non-permissible values?

a) $\dfrac{2y^2+y-10}{y^2+3y-10}$

$= \dfrac{(2y+5)(y-2)}{(y+5)(y-2)}$ $y \neq 2, -5$

$= \boxed{\dfrac{2y+5}{y+5}} \;, y \neq 2, -5$

b) $\dfrac{6-2m}{m^2-9}$

$= \dfrac{-2(-3+m)}{(m+3)(m-3)}$ $m \neq -3, 3$

$= \boxed{\dfrac{-2}{m+3}} \;, m \neq -3, 3$

Ex.3:   Consider the expression $\dfrac{16x^2-9y^2}{8x-6y}$.

a) What expression represents the non-permissible values for x?
b) What is the non-permissible value for y if $x = 3$?
c) Simplify the rational expression.
d) Evaluate the expression for $x = 2.6$ and $y = 1.2$. Show two ways to determine the answer.

a) $8x - 6y \neq 0$

$8x \neq 6y$

$x \neq \dfrac{6y}{8} = \dfrac{3y}{4}$

$\boxed{x \neq \dfrac{3y}{4}}$

b) $8x - 6y \neq 0$

$8(3) - 6y \neq 0$

$24 \neq 6y$

$\dfrac{24}{6} \neq \dfrac{6y}{6}$

$\boxed{y \neq 4}$

c) $\dfrac{16x^2-9y^2}{8x-6y}$ ← Diff. of sq.

← GCF=2

$= \dfrac{(4x+3y)(4x-3y)}{2(4x-3y)}$

$= \boxed{\dfrac{4x+3y}{2}}$

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6.2: Multiplying and Dividing Rational Expressions

Objectives:
- determine the product or quotient of rational expressions in simplest form

Multiplying Rational Expressions
Just like multiplying fractions! <mark>Include npvs!</mark>

Factor first!

* multiply **top to top** and **bottom to bottom**!

$$\left(\frac{5}{8}\right)\left(\frac{4}{15}\right) = \frac{(5)(4)}{(8)(15)} = \frac{(5)(4)}{(2)(4)(3)(5)}$$
$$= \frac{(5)(4)}{2(4)(3)(5)} = \frac{1}{6}$$

$$\left(\frac{4x^2}{3xy}\right)\left(\frac{y^2}{8x}\right) = \frac{(4x^2)(y^2)}{(3xy)(8x)}$$
$$= \frac{4x^2y^2}{24x^2y} = \frac{y}{6}, \; x \neq 0, \; y \neq 0$$

Ex.1: Multiply. Write your answer in simplest form. Identify all non-permissible values first.

a) $\dfrac{a^2 - a - 12}{a^2 - 9} \times \dfrac{a^2 - 4a + 3}{a^2 - 4a}$ **FACTOR**

b) $\dfrac{y^2 - 9}{r^3 - r} \times \dfrac{r^2 - r}{y + 3}$ **FACTOR**

a) FACTOR

$$= \frac{(a-4)(a+3)}{(a-3)(a+3)} \times \frac{(a-3)(a-1)}{a(a-4)}$$

npv's: $a \neq 3, -3, 0, 4$

$$= \boxed{\frac{a-1}{a}}$$

b)

$$= \frac{(y+3)(y-3)}{r(r^2-1)} \times \frac{r(r-1)}{y+3}$$

$$= \frac{(y+3)(y-3)}{r(r+1)(r-1)} \times \frac{r(r-1)}{y+3}$$

npv's: $r \neq 0, -1, 1$
$y \neq -3$

$$= \boxed{\frac{y-3}{r+1}}$$

Dividing Rational Expressions
Just like dividing fractions! Include npvs!

XOXO $\times \circledS$

* *Multiply by the Reciprocal* **(or KISS AND FLIP!)**

$$\frac{5}{3} \div \frac{1}{6} = \frac{5}{3} \otimes \frac{6}{1}$$
$$= 10$$

$= \dfrac{\frac{5}{3}}{\frac{1}{6}}$

$$\frac{3x^2}{y^2} \div \frac{x}{y} = \frac{3x^2}{y^2} \otimes \frac{y}{x}$$
$$= \frac{3x}{y}, \; x \neq 0, \; y \neq 0$$

$\dfrac{\frac{3x^2}{y^2}}{\frac{x}{y}}$

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Ex.3:   Determine the quotient in simplest form. Identify all non-permissible values.

FACTOR

a) $\dfrac{x^2 - 4}{x^2 - 4x} \div \dfrac{x^2 + x - 6}{x^2 + x - 20}$

$= \dfrac{(x+2)(x-2)}{x(x-4)} \div \dfrac{(x+3)(x-2)}{(x+5)(x-4)}$

npv's: $x \neq 0, 4, -5, -3, 2$

$= \dfrac{(x+2)\cancel{(x-2)}}{x\cancel{(x-4)}} \times \dfrac{(x+5)\cancel{(x-4)}}{(x+3)\cancel{(x-2)}} = \boxed{\dfrac{(x+2)(x+5)}{x(x+3)}}$

new npv's!

FACTOR

b) $\dfrac{c^2 - 6c - 7}{c^2 - 49} \div \dfrac{c^2 + 8c + 7}{c^2 + 7c}$

$= \dfrac{\cancel{(c-7)}(c+1)}{(c+7)\cancel{(c-7)}} \div \dfrac{\cancel{(c+7)}(c+1)}{c\cancel{(c+7)}}$

npv's: $c \neq 0, 7, -7, -1$

$= \dfrac{\cancel{c+1}}{c+7} \times \dfrac{c}{\cancel{c+1}} = \boxed{\dfrac{c}{c+7}}$

Ex.3: Simplify. What are the non-permissible values?

FACTOR

$\overset{-30}{2m^2 - 7m - 15}$

$2m^2 \overset{\wedge}{-10m + 3m - 15}$

$2m(m-5) + 3(m-5)$

a) $\dfrac{2m^2 - 7m - 15}{2m^2 - 10m} \div \dfrac{4m^2 - 9}{6} \times (3 - 2m)$

$= \dfrac{(m-5)(2m+3)}{2m(m-5)} \div \dfrac{(2m+3)(2m-3)}{6} \times (3-2m)$

npv's: $m \neq 0, 5, -\dfrac{3}{2}, \dfrac{3}{2}$

$= \dfrac{\cancel{2m+3}}{2m} \times \dfrac{6}{\cancel{(2m+3)}\cancel{(2m-3)}} \times \overset{-}{(3-2m)} = \dfrac{-6}{2m} = \boxed{\dfrac{-3}{m}}$

new npv's

$\overset{-36}{3x^2 - 5x - 12}$

$3x^2 \overset{\wedge}{-9x + 4x - 12}$

$3x(x-3) + 4(x-3)$

b) $\dfrac{3x + 12}{3x^2 - 5x - 12} \div \dfrac{12}{3x + 4} \times \dfrac{2x - 6}{x + 4}$

$= \dfrac{3(x+4)}{(x-3)(3x+4)} \div \dfrac{12}{3x+4} \times \dfrac{2(x-3)}{x+4}$

npv's: $x \neq 3, \dfrac{4}{3}, -4$

$= \dfrac{3\cancel{(x+4)}}{\cancel{(x-3)}\cancel{(3x+4)}} \times \dfrac{\cancel{3x+4}}{12} \times \dfrac{2\cancel{(x-3)}}{\cancel{x+4}}$

$= \dfrac{3 \times 2}{12} = \dfrac{6}{12} = \boxed{\dfrac{1}{2}}$

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6.3: Adding and Subtracting Rational Expressions

Objectives: (+)   (-)
- Find the sum or difference of rational expressions

Adding or Subtracting Rational Expressions

To add or subtract rational expressions, follow procedures **_similar_** to those used in adding or subtracting rational numbers.   *need common denominator
*add/subtract numerators!

$$\frac{5}{12} + \frac{2}{12} = \frac{7}{12}$$

eg. $\frac{1}{4} + \frac{3}{2} \cdot \frac{\times 2}{\times 2} = \frac{1}{4} + \frac{6}{4} = \frac{7}{4}$

Similarly, rational expressions with the same denominator are alike, and you can group them together. (COMMON DENOMINATORS!)

$$\frac{5x}{x+12} + \frac{2x-3}{x+12} = \frac{5x + (2x-3)}{x+12}$$
$$= \frac{7x-3}{x+12}$$

Add (or subtract) the numerators. The denominator does not change.

*FACTOR first! *state npv's!

If rational expressions have different denominators, rewrite them as equivalent expressions with the least common denominator (LCD).

Ex.1:   Simplify and identify the non-permissive values of the variables.

a) $\dfrac{10a+5}{ab} - \dfrac{(3a-2)}{ab}$ npv's:
$a \neq 0$
$b \neq 0$
LCD = ab

$= \dfrac{10a+5-(3a-2)}{ab}$

$= \boxed{\dfrac{7a+7}{ab}}$

b) $\dfrac{2x+4}{x^2-9} + \dfrac{7}{x-3}$ FACTOR
npv's:
$x \neq -3, 3$

$= \dfrac{2x+4}{(x+3)(x-3)} + \dfrac{7(x+3)}{x-3(x+3)}$ LCD = $(x+3)(x-3)$

$= \dfrac{2x+4 + 7(x+3)}{(x+3)(x-3)} = \boxed{\dfrac{9x+25}{(x+3)(x-3)}}$

Ex.2:   Identify the least common denominator for each group of rational expressions.

a) $\dfrac{1}{2x}, \dfrac{1}{3x}, \dfrac{1}{y}$

LCD = $2x \cdot 3 \cdot y$
$= 6xy$

b) $\dfrac{1}{x+5}, \dfrac{1}{(x+5)(x-4)}, \dfrac{1}{(x-4)(x+7)}$

LCD = $(x+5)(x-4)(x+7)$

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c) $\dfrac{1}{x^2 + 6x + 9}, \dfrac{1}{x^2 + 8x + 15}$

factor first!

$\dfrac{1}{(x+3)(x+3)}, \dfrac{1}{(x+5)(x+3)}$

$LCD = (x+3)(x+3)(x+5)$

☆ Don't Expand...

Ex.3: Simplify. Express your answers in simplest form.

a) $\dfrac{2x}{xy} + \dfrac{4}{x^2} - 3, x \neq 0, y \neq 0$

$LCD = x^2y$

$\dfrac{2x \cdot x}{xy \cdot x} + \dfrac{4 \cdot y}{x^2 \cdot y} - \dfrac{3x^2y}{x^2y} = \boxed{\dfrac{2x^2 + 4y - 3x^2y}{x^2y}}$

can't cancel unless factored!

FACTOR!

b) $\dfrac{y^2 - 20}{y^2 - 4} + \dfrac{y-2}{y+2}, y \neq \pm 2$

$= \dfrac{y^2 - 20}{(y+2)(y-2)} + \dfrac{(y-2)(y-2)}{(y+2)(y-2)}$ $LCD = (y+2)(y-2)$ $= \dfrac{y^2 - 20 + y^2 - 2y - 2y + 4}{(y+2)(y-2)}$

$= \dfrac{y^2 - 20 + (y-2)(y-2)}{(y+2)(y-2)}$ $= 2y^2 - 4 - 16 \leftarrow$ FACTOR! $2) \quad -8)$

$\quad 2)$

Thurs.

c) $\dfrac{1 + \dfrac{1}{x}}{x - \dfrac{1}{x}}, x \neq 0, x \neq \pm 1$

$x \cdot \dfrac{1}{x} + \dfrac{1}{x} = \dfrac{x+1}{x}$

$x \cdot \dfrac{x}{x} - \dfrac{1}{x} = \dfrac{x^2 - 1}{x}$

÷ Fractions → X by Reciprocal!

$= \dfrac{x+1}{x} \times \dfrac{x}{x^2 - 1} = \dfrac{x+1}{(x+1)(x-1)} = \boxed{\dfrac{1}{x-1}}$

Your Turn Simplify. What are the non-permissible values?

a) $\dfrac{4}{p^2 - 1} + \dfrac{3}{p+1} = \dfrac{4}{(p+1)(p-1)} + \dfrac{3(p-1)}{(p+1)(p-1)}$

npv's: $p \neq -1, 1$

$LCD = (p+1)(p-1)$

$= \dfrac{4 + 3(p-1)}{(p+1)(p-1)} = \dfrac{4 + 3p - 3}{(p+1)(p-1)} = \boxed{\dfrac{3p+1}{(p+1)(p-1)}}$

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b) $\dfrac{2-\dfrac{4}{y}}{y-\dfrac{4}{y}}$

$\rightarrow \dfrac{y \cdot 2}{y} - \dfrac{4}{y} = \dfrac{2y-4}{y}$

$\rightarrow \dfrac{y \cdot y}{y} - \dfrac{4}{y} = \dfrac{y^2-4}{y}$

$= \dfrac{2y-4}{y} \times \dfrac{y}{y^2-4}$

npv's: $y \neq 0, 2, -2$

$= \dfrac{2(y-2)}{y} \times \dfrac{y}{(y+2)(y-2)} = \boxed{\dfrac{2}{y+2}}$

6.4: Rational Equations

Objectives:
- Determine the solution to a rational equation algebraically

Solving Rational Equations!
1. Factor each denominator
2. Identify the non-permissible values
3. Multiply both sides of the equation by the lowest common denominator
4. Solve by isolating the variable on one side of the equation
5. Check your answers $\ast \rightarrow$ Reject any extraneous roots!

Why? Cancel Fractions!

$2\left(\dfrac{1}{2}x = 4\right)$

$x = 8$

Ex.1: Solve the following equation. What values are non-permissible?

$4x\left(\dfrac{x}{4} - \dfrac{7}{x} = 3\right)$

npv's: $x \neq 0$
LCD $= 4x$

$4x\left(\dfrac{x}{4}\right) - 4x\left(\dfrac{7}{x}\right) = 4x(3)$

$x^2 - 28 = 12x$

$x^2 - 12x - 28 = 0$

$(x-14)(x+2) = 0$

$x = -14, 2$

Check:

$x \neq 0, \quad x = 14, -2$

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Ex.2: Solve the following equation. What values are non-permissible?

$$\frac{2}{z^2 - 4} + \frac{10}{6z + 12} = \frac{1}{z - 2}$$

npv's: $z \neq -2, 2$

LCD $= (z+2)(z-2)3$

FACTOR!

$$3(z+2)(z-2)\left(\frac{2}{(z+2)(z-2)} + \frac{\overset{5/3}{10}}{6(z+2)} = \frac{1}{z-2} \right)$$

$$6 + 5(z-2) = 3(z+2)$$
$$6 + 5z - 10 = 3z + 6$$
$$2z - 4 = 6$$

$2z = 10$
$z = 5$

check

$\boxed{z=5,}\ z \neq -2, 2$

Your Turn Solve. What are the non-permissible values?

$$\frac{3x}{x + 2} - \frac{5}{x - 3} = \frac{-25}{x^2 - x - 6}$$

npv's: $x \neq -2, 3$

LCD $= (x+2)(x-3)$

$$(x+2)(x-3)\left(\frac{3x}{x+2} - \frac{5}{x-3} = \frac{-25}{(x-3)(x+2)} \right)$$

$$3x(x-3) - 5(x+2) = -25$$
$$3x^2 - 9x - 5x - 10 = -25$$
$$3x^2 - 14x + 15 = 0$$

$$3x^2 - 14x + 15 = 0$$
$$\overset{45}{3x^2 - 9x - 5x + 15} = 0$$
$$3x(x-3) - 5(x-3) = 0$$
$$(x-3)(3x-5) = 0$$

$x = \cancel{3}\ 5/3$ → Check! $x \neq 3, -2$

only $\boxed{x = 5/3}$

Ex.3: Two friends share a paper route. Sheena can deliver the papers in 40 min. Jeff can cover the same route in 50 min. How long, to the nearest minute, does the paper route take if they work together?

RATE

$$\text{Rate} = \frac{\text{Job}}{\text{Time}}$$

Sheena: $R_S = \dfrac{1 \text{ route}}{40 \text{ min.}}$

Jeff: $R_J = \dfrac{1 \text{ route}}{50 \text{ min.}}$

Combined: $R_S + R_J$

$$\frac{1}{40} + \frac{1}{50} = \frac{5}{200} + \frac{4}{200} = \frac{9}{200}$$ routes / minutes

$$\text{Time} = \frac{\text{Job}}{\text{Rate}} = \frac{1}{\frac{9}{200}} = \frac{200}{9} = \boxed{22.2 \text{ minutes}}$$

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Ex.4: Andrea and Phary are sharing a bag of popcorn at the movies. By himself, Phary can eat the whole bag of popcorn in 20 min. Andrea takes 25 min to eat the whole bag. If they both eat popcorn at their usual rates, how quickly will they eat the popcorn?

RATE

$$Rate = \frac{Job}{Time}$$

Combined: $\dfrac{1}{20} + \dfrac{1}{25} = \dfrac{5}{100} + \dfrac{4}{100} = \dfrac{9}{100}$ ← Bags ← Minutes

$$Time = \frac{Job}{Rate} = \frac{1}{\frac{9}{100}} = \frac{100}{9} = \boxed{11.1 \text{ minutes}}$$

Ex.5: A group of friends go on a 3-h bike ride together. They ride 15 km with the wind at their backs, and then 15 km straight into the wind. The wind adds or subtracts 3 km/h from their speed. What is the average speed of the group of friends with no wind?

Distance Speed Time

	Distance	Speed	Time
with wind	15 km	$x+3$	$\dfrac{15}{x+3}$
against wind	15 km	$x-3$	$\dfrac{15}{x-3}$

$\Big\} = 3$

let x = average speed with no wind.

$Speed = \dfrac{Distance}{Time}$

$Time = \dfrac{Distance}{Speed}$

$$(x+3)(x-3)\left(\frac{15}{x+3} + \frac{15}{x-3} = 3\right)$$

npv's: $x \neq -3, 3$
LCD = $(x+3)(x-3)$

$$15(x-3) + 15(x+3) = 3(x+3)(x-3)$$
$$15x - \cancel{45} + 15x + \cancel{45} = 3(x^2 - \cancel{3x} + \cancel{3x} - 9)$$
$$30x = 3x^2 - 27$$
$$0 = 3x^2 - 30x - 27$$
$$0 = x^2 - 10x - 9 \quad \leftarrow \text{doesn't factor...}$$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(-9)}}{2(1)}$$

speed can't be negative

$$= \frac{10 \pm \sqrt{136}}{2} = 10.8, \; {}^{-}0.8$$

$$\boxed{Average \; speed = 10.8 \; km/h}$$

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